# Lie algebra automorphisms as Lie point symmetries and the solution space for Bianchi Type $I, II, IV, V$ vacuum geometries

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**Abstract**

Lie group symmetry analysis for systems of coupled, nonlinear ordinary differential equations is performed in order to obtain the entire solution space to Einstein's field equations for vacuum Bianchi spacetime geometries. The symmetries used are the automorphisms of the Lie algebra of the corresponding three-dimensional isometry group acting on the hyper-surfaces of simultaneity for each Bianchi Type, as well as the scaling and the time reparametrization symmetry. The method is applied to Bianchi Types $I, II, IV$ and $V$. The result is the acquisition, in each case, of the entire solution space of either Lorenzian of Euclidean signature. This includes all the known solutions for each Type and the general solution of Type $IV$ (in terms of sixth Painlevé transcendent $\mathbf{P_{VI}}$).

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# 1 Introduction

The exploitation of the group of automorphisms in order to obtain a unified treatment of spatially homogeneous Bianchi spacetime geometries has a rather long history as it goes back to the early 1960's [1]. In 1979, Harvey [2] found the automorphisms of all three-dimensional Lie algebras, while the corresponding results for the four-dimensional Lie algebras have been presented in [3]. In Jantzen's tangent space approach the automorphism matrices are considered as the means for achieving a convenient parametrization of a full scale factor matrix in terms of a desired, diagonal matrix [4, 5, 6]. Siklos used these time-dependent automorphisms as a tool for the proper choice of variables aiming at a simplification of the ensuing equations [7], while Samuel and Ashtekar were the first to look upon automorphisms from a space viewpoint [8]. The notion of *Time-Dependent Automorphism Inducing Diffeomorphisms* ,i.e., coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse, and the shift vector has been developed in [9]. The use of these covariances enables one to set the shift vector to zero without destroying manifest spatial homogeneity. At this stage one can use the rigid automorphisms, i.e., the remaining gauge symmetry, as Lie point symmetries of Einstein's field equations in order to reduce the order of these equations and ultimately completely integrate them [10].

The methodology applied in the present work is the Lie group symmetry analysis of differential equations. For a detailed account of this approach see, e.g., [11], [12], [13]. Just to put it simply, a symmetry of a system of differential equations is a transformation mapping of any solution of the system to another solution. In other words, the symmetry group of the system is the group transforming solutions of the system to other solutions. In fact, it is the largest local group of transformations acting on the system variables. Such groups are Lie groups depending on continuous parameters and consisting of either point transformations, acting on the systems space of independent and dependent variables or, more generally, contact transformations that act also on all first derivatives of the dependent variables. The theory can be also generalized to the so called higher-order (*Lie-Backlund*) symmetries, by including in the transformation of the independent variables derivatives of the dependent. In this work we use all these three kinds. In general, the aforementioned transformations are nonlinear and the main benefit of the Lie group symmetry analysis comes from the replacement of the nonlinear symmetry conditions of the system investigated by "linear" conditions associated with the infinitesimal generators of the symmetries. To this purpose, it is necessary first to determine a class of general admissible variable transformations and then to search for special members of this class under which the system of differential equations remains invariant. It is rather evident, that the degree of generality of the admissible transformations is proportional to the number of the existing symmetries. One should further stress, that one of the main ingredients of the method lies on the notion of "prolongation" of a group action on the space of derivatives of the systems dependent variables up to any finite order. Thus one is able to deal with differential equations of any order. In closing this short introduction, we must point out that the knowledge of a symmetry group of a

higher order differential equation has much the same consequences as the knowledge of a symmetry group of the corresponding system of first order differential equations. In fact, we will make use of the following two important theorems (for a proof, see [12]):

**Theorem 1.1** *Let*

$$\frac{dy^{\nu}}{dx} = F_{\nu}(x, y), \quad \nu = 1, \ldots, q \tag{1.1}$$

*be a first order system of $q$ differential equations and suppose that $G$ is a one-parameter symmetry group of the system. Then, there exists a change of variables $(t, u) = \psi(x, y)$ under which the system can be written as*

$$\frac{du^{\nu}}{dt} = H_{\nu}\left(t,\, u^1, \ldots,\, u^{q-1}\right), \quad \nu = 1, \ldots, q \tag{1.2}$$

*so that the original system is reduced to a new system of $q-1$ differential equations for $u^1, u^2, , u^{q-1}$ together with the integral*

$$u^q(t) = \int H_q\left(t,\, u^1(t), \ldots,\, u^{q-1}(t)\right) dt + c. \tag{1.3}$$

**Theorem 1.2** *If the system of $q$ first order differential equations considered in* **Theorem 1.1** *admits an $r$-parameter solvable symmetry group, then the solution of the system can be found by quadrature from the solution of a reduced system of $q-r$ first order differential equations. If the original system is invariant under a $q$-parameter solvable symmetry group, then its general solution can be found by quadrature alone.*

The paper is organized as follows: in Section 2, we give our method. In Sections 3-6, the application of the method to Bianchi Types I, II, V and IV is presented. Finally, some discussion and concluding remarks are given in Section 7.

# 2 The Method

It is known, that the line element for spatially homogeneous spacetime geometries with a simply transitive action of the corresponding isometry group [14], [15], assumes the form

$$ds^2 = \left(N^{\alpha}N_{\alpha} - N^2\right)dt^2 + 2N_{\alpha}\sigma^{\alpha}_i dx^i dt + \gamma_{\alpha\beta}\sigma^{\alpha}_i\sigma^{\beta}_j dx^i dx^j \tag{2.1}$$

with the 1-forms $\boldsymbol{\sigma}^{\alpha}$ defined from

$$d\boldsymbol{\sigma}^{\alpha} = C^{\alpha}_{\beta\gamma}\boldsymbol{\sigma}^{\beta} \wedge \boldsymbol{\sigma}^{\gamma} \Leftrightarrow \sigma^{\alpha}_{i,j} - \sigma^{\alpha}_{j,i} = 2C^{\alpha}_{\beta\gamma}\sigma^{\gamma}_i\sigma^{\beta}_j. \tag{2.2}$$

(Small Latin letters denote world space indices while small Greek letters count the different basis one-forms; both types of indices range over the values of 1, 2, 3).

Then the Einstein field equations are (see, e.g.[16], [9]):

$$E_o \doteq K^{\alpha\beta}K_{\alpha\beta} - K^2 - \mathbf{R} = 0 \tag{2.3}$$

$$E_\alpha \doteq K^\mu_\alpha C^\epsilon_{\mu\epsilon} - K^\mu_\epsilon C^\epsilon_{\alpha\mu} = 0 \tag{2.4}$$

$$\begin{aligned} E_{\alpha\beta} \doteq \dot{K}_{\alpha\beta} + N\left(2K^\tau_\alpha K_{\tau\beta} - K K_{\alpha\beta}\right) + \\ + 2N^\rho\left(K_{\alpha\nu}C^\nu_{\beta\rho} + K_{\beta\nu}C^\nu_{\alpha\rho}\right) - N\mathbf{R}_{\alpha\beta} = 0 \end{aligned} \tag{2.5}$$

where

$$K_{\alpha\beta} = -\frac{1}{2N}\left(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu}C^\nu_{\beta\rho}N^\rho + 2\gamma_{\beta\nu}C^\nu_{\alpha\rho}N^\rho\right) \tag{2.6}$$

is the extrinsic curvature of the three-dimensional hypersurface and

$$\begin{aligned} \mathbf{R}_{\alpha\beta} = \quad & C^\kappa_{\sigma\tau}C^\lambda_{\mu\nu}\gamma_{\kappa\alpha}\gamma_{\beta\lambda}\gamma^{\sigma\nu}\gamma^{\tau\mu} + 2C^\kappa_{\lambda\beta}C^\lambda_{\alpha\kappa} + 2C^\mu_{\kappa\alpha}C^\nu_{\beta\lambda}\gamma_{\mu\nu}\gamma^{\kappa\lambda} + \\ & 2C^\lambda_{\kappa\beta}C^\mu_{\mu\nu}\gamma_{\alpha\lambda}\gamma^{\kappa\nu} + 2C^\lambda_{\kappa\alpha}C^\mu_{\mu\nu}\gamma_{\beta\lambda}\gamma^{\kappa\nu} \end{aligned} \tag{2.7}$$

is the Ricci tensor.

In [9], particular spacetime coordinate transformations have been found, which reveal as symmetries of (2.3), (2.4), and (2.5) the following induced transformations of the dependent variables $N$, $N_\alpha$, $\gamma_{\alpha\beta}$:

$$\tilde{N} = N, \quad \tilde{N}_\alpha = \Lambda^\rho_\alpha\left(N_\rho + \gamma_{\rho\sigma}P^\sigma\right), \quad \tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu\Lambda^\beta_\nu\gamma_{\alpha\beta} \tag{2.8}$$

where the matrix $\Lambda$ and the triplet $P^\alpha$ must satisfy:

$$\Lambda^\alpha_\rho C^\rho_{\beta\gamma} = C^\alpha_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta \tag{2.9a}$$

$$2P^\mu C^\alpha_{\mu\nu}\Lambda^\nu_\beta = \dot{\Lambda}^\alpha_\beta \tag{2.9b}$$

These transformations were first presented in [4], see also the discussion on p. 3586 of [9].

For all Bianchi Types, this system of equations admits solutions that contain three arbitrary functions of time plus several constants depending on the automorphism group of each Type. The three functions of time are distributed among $\Lambda$ and $P^\alpha$ (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the scale factor matrix or to set the shift vector $N^\alpha$ to zero. The second action can always be taken, since, for every Bianchi Type, all three functions appear in $P^\alpha$.

In this work we adopt the latter point of view. Having used the freedom stemming from the three arbitrary functions in order to set the shift vector to zero, there is still a remaining "gauge" freedom consisting of constant $\Lambda^\alpha_\beta$ (automorphism group matrices of the Lie group defined by the structure constants $C^\alpha_{\beta\gamma}$. Indeed, the system (2.9) accepts the solution $\Lambda^\alpha_\beta =$ const., $P^\alpha = \mathbf{0}$. The latter are also known in the literature as rigid symmetries [17].

The generators of the corresponding motions $\tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu\Lambda^\beta_\nu\gamma_{\alpha\beta}$ induced in the space of the dependent variables spanned by $\gamma_{\alpha\beta}$'s (the lapse is given in terms of $\gamma_{\alpha\beta}$, $\dot{\gamma}_{\alpha\beta}$ by algebraically solving the quadratic constraint equation), are:

$$X_I = \lambda^\rho_{I\alpha}\gamma_{\rho\beta}\frac{\partial}{\partial\gamma_{\alpha\beta}} \tag{2.10}$$

with $\lambda^{\alpha}_{\beta}$ satisfying

$$\lambda^{\alpha}_{I\rho}\, C^{\rho}_{\beta\gamma} = \lambda^{\rho}_{I\beta}\, C^{\alpha}_{\rho\gamma} + \lambda^{\rho}_{I\gamma}\, C^{\alpha}_{\beta\rho}. \tag{2.11}$$

These generators define a Lie algebra and each one of them induces, through its integral curves, a transformation on the configuration space spanned by the $\gamma_{\alpha\beta}$'s [18]. If a generator is brought to its normal form (i.e., $\frac{\partial}{\partial z_i}$), then the Einstein field equations, written in terms of the new dependent variables, will not explicitly involve $z_i$. They thus become a first order system in the function $\dot{z}_i$ [11]. If the aforesaid Lie algebra is abelian, then all generators can be brought to their normal form simultaneously. If the Lie algebra is non-abelian, then we can diagonalize in one step those generators corresponding to any eventual abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of the field equations if the Lie algebra of the $X_{(I)}$'s is solvable [12]. One can thus repeat the previous step by choosing one of these remaining generators and bring it to its normal form. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally, if the Lie algebra does not contain any abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system of equations, and search for its symmetries (if there are any). Lastly, two further symmetries of (2.3), (2.4), and (2.5) are also present and can be used in conjunction with the constant automorphisms: The time reparametrization $t \to t + \alpha$, owing to the non-appearance of time in these equations (the system being autonomous), and the scaling by a constant $\gamma_{\alpha\beta} \to \lambda\gamma_{\alpha\beta}$ (homothety) as can be straightforwardly verified. Hence, in every Bianchi Type there are, added to the $X_{(I)}$ generators, also the following generators:

$$Y_1 = \frac{\partial}{\partial t} \tag{2.12}$$

$$Y_2 = \gamma_{11}\frac{\partial}{\partial\gamma_{11}} + \gamma_{12}\frac{\partial}{\partial\gamma_{12}} + \gamma_{13}\frac{\partial}{\partial\gamma_{13}} + \gamma_{22}\frac{\partial}{\partial\gamma_{22}} + \gamma_{23}\frac{\partial}{\partial\gamma_{23}} + \gamma_{33}\frac{\partial}{\partial\gamma_{33}} \tag{2.13}$$

These generators commute among themselves, as well as with the $X_{(I)}$'s:

$$[X_I, Y_\alpha] = 0 \qquad \{I = 1, 2, 3, 4 \,|\, \alpha = 1, 2\} \tag{2.14}$$

# 3 Bianchi Type I

In the Bianchi Type I model, the structure constants are

$$C^{\alpha}{}_{\beta\gamma} = 0 \qquad \text{for every value of} \quad \alpha,\, \beta,\, \gamma \tag{3.1}$$

By using this relation and the definition of the 1-forms $\boldsymbol{\sigma}^{\alpha}$ (2.2), we get

$$\boldsymbol{\sigma}^1 = \mathbf{d}\, x, \quad \boldsymbol{\sigma}^2 = \mathbf{d}\, y, \quad \boldsymbol{\sigma}^3 = \mathbf{d}\, z \tag{3.2}$$

with the corresponding Killing fields

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z \tag{3.3}$$

From equations (2.9) we have that the triplet $P^\alpha(t)$ is arbitrary while the automorphisms $\Lambda^\alpha_\beta$ are constant, i.e. they are elements of the general linear group $GL_3(\mathbb{R})$. Hence, the symmetry group of the field equations is a ten-parameter group consisting of the nine variables of the $GL_3(\mathbb{R})$ group plus the symmetry produced by the generator (2.12), while the generator (2.13) is already included in the $GL_3(\mathbb{R})$ group. Using the triplet $P^\alpha(t)$, we can set the shift vector $N^\alpha$ to zero by applying (2.8). Furthermore, in order to simplify the field equations, we choose the time gauge $N^2 = \gamma$, $\gamma = \det(\gamma_{\alpha\beta})$. In this case, eq. (2.4) is satisfied identically, while equations (2.3), (2.5) take the form

$$\gamma^{\mu\alpha}\,\gamma^{\nu\beta}\,\dot{\gamma}_{\alpha\beta}\,\dot{\gamma}_{\mu\nu} - \gamma^{\alpha\beta}\,\gamma^{\mu\nu}\,\dot{\gamma}_{\alpha\beta}\,\dot{\gamma}_{\mu\nu} = 0 \tag{3.4}$$

$$\ddot{\gamma}_{\alpha\beta} - \gamma^{\rho\tau}\,\dot{\gamma}_{\rho\alpha}\,\dot{\gamma}_{\tau\beta} = 0 \tag{3.5}$$

The last equation can be integrated if multiplied by $\gamma^{\alpha\sigma}$:

$$\begin{aligned}
\gamma^{\alpha\sigma}\,\ddot{\gamma}_{\alpha\beta} - \gamma^{\alpha\sigma}\,\gamma^{\rho\tau}\,\dot{\gamma}_{\rho\alpha}\,\dot{\gamma}_{\tau\beta} &= 0 \Rightarrow \\
\gamma^{\alpha\sigma}\,\ddot{\gamma}_{\alpha\beta} + \dot{\gamma}^{\alpha\sigma}\,\gamma^{\rho\tau}\,\gamma_{\rho\alpha}\,\dot{\gamma}_{\tau\beta} &= 0 \Rightarrow \\
\gamma^{\alpha\sigma}\,\ddot{\gamma}_{\alpha\beta} + \dot{\gamma}^{\alpha\sigma}\,\dot{\gamma}_{\alpha\beta} &= 0 \Rightarrow \\
\frac{d}{dt}(\gamma^{\alpha\sigma}\,\dot{\gamma}_{\alpha\beta}) &= 0 \Rightarrow \\
\gamma^{\alpha\sigma}\,\dot{\gamma}_{\alpha\beta} = \vartheta^\sigma{}_\beta, \quad \vartheta^\sigma{}_\beta &= \text{const.}
\end{aligned} \tag{3.6}$$

Multiplying the latter by $\gamma_{\sigma\rho}$ we obtain the linear system

$$\dot{\gamma}_{\beta\rho} = \vartheta^\sigma{}_\beta\,\gamma_{\sigma\rho} \quad \text{or, in matrix form,} \quad \dot{\gamma} = \vartheta^T\,\gamma \tag{3.7}$$

Thus, the quadratic constraint (3.4) can be written as a relation between the constant elements of the matrix $\vartheta$

$$\vartheta^\alpha{}_\beta\vartheta^\beta{}_\alpha - (\vartheta^\alpha{}_\alpha)^2 = 0 \tag{3.8}$$

The general solution of the linear system (3.7) is of the form

$$\gamma_{\alpha\beta} = \exp{(\vartheta^T\,t)^\rho}_\alpha\,c_{\rho\beta}, \quad \exp{(\vartheta^T\,t)^\rho}_\alpha\,c_{\rho\beta} = \exp{(\vartheta^T\,t)^\rho}_\beta\,c_{\rho\alpha}, \quad c_{\alpha\beta} = \text{const.} \tag{3.9}$$

where the symmetry of $\dot{\gamma}_{\alpha\beta}$ has been taken into consideration.

In order to find the analytic form of this solution, it is necessary to calculate the matrix exponential of the $3 \times 3$ matrix $\vartheta$. To this purpose, we first note that $\gamma_{\alpha\beta}$ transforms, under $GL_3(\mathbb{R})$, as

$$\gamma_{\alpha\beta} = \Lambda^\mu{}_\alpha\,\Lambda^\nu{}_\beta\,\widetilde{\gamma}_{\mu\nu} \Rightarrow \gamma = \Lambda^T\,\widetilde{\gamma}\,\Lambda, \quad \Lambda \in GL_3(\mathbb{R}) \tag{3.10}$$

so we have for the linear system (3.7)

$$\Lambda^T\,\dot{\widetilde{\gamma}}\,\Lambda = \vartheta^T\,\Lambda^T\,\widetilde{\gamma}\,\Lambda \Rightarrow \dot{\widetilde{\gamma}} = \widetilde{\vartheta}\,\widetilde{\gamma}, \qquad \widetilde{\vartheta} = \Lambda^{-T}\,\vartheta^T\,\Lambda^T \tag{3.11}$$

Now, $\Lambda$ is an arbitrary matrix, hence it can be used to simplify the matrix $\vartheta$. The degree of simplification depends on the eigenvalues of the matrix $\vartheta$ which may be (i) three real, distinct eigenvalues, (ii) one real and two complex conjugate eigenvalues, (iii) three real eigenvalues two of which are repeated and, (iv) three real, repeated eigenvalues. If the matrix $\vartheta$ is the zero matrix, then $\gamma_{\alpha\beta} = $ const. and the metric thus obtained describes Minkowski flat spacetime .

## 3.1 Three real distinct eigenvalues

In this case, we can choose the matrix $\Lambda$ such as to diagonalize the matrix $\vartheta$, i.e.

$$\widetilde{\vartheta} = diag(p_1, p_2, p_3) \Rightarrow \exp\widetilde{\vartheta}\, t = diag(e^{p_1\, t}, e^{p_2\, t}, e^{p_3\, t}). \tag{3.12}$$

As a result, the solution obtained from (3.9) is

$$\widetilde{\gamma} = diag(c_1\, e^{p_1\, t},\ c_2\, e^{p_2\, t},\ c_3\, e^{p_3\, t}) \tag{3.13}$$

and the constants $(p_1, p_2, p_3)$ are related through (3.8), i.e.

$$p_1^2 + p_2^2 + p_3^2 = (p_1 + p_2 + p_3)^2 \Rightarrow p_1\, p_2 + p_1\, p_3 + p_2\, p_3 = 0 \tag{3.14}$$

Since the constants $(p_1, p_2, p_3)$ are different, none of them vanishes. Thus, dividing by, let's say, $p_1^2$, we can eliminate it, so

$$\frac{p_2}{p_1} + \frac{p_3}{p_1} + \frac{p_2}{p_1}\frac{p_3}{p_1} = 0 \Rightarrow \alpha + \beta + \alpha\,\beta = 0, \quad \alpha = \frac{p_2}{p_1},\ \beta = \frac{p_3}{p_1} \tag{3.15}$$

The coefficients of the diagonal elements of the matrix $\widetilde{\gamma}$ can, by a proper coordinate transformation of $(x, y, z)$, become the same and equal to $p_1^{-2}$. By putting $t = p_1^{-1}\,\tau$, we get the following final form of the metric:

$$d\,s^2 = -e^{(1+\alpha+\beta)\,\tau}\, d\,\tau^2 + e^{\tau}\, d\,x^2 + e^{\alpha\,\tau}\, d\,y^2 + e^{\beta\,\tau}\, d\,z^2 \tag{3.16}$$

where the constants $(\alpha, \beta)$ satisfy (3.15). The multiplicative constant $p_1^{-2}$ does not appear in the metric due to the homothety

$$H = 2\,(\alpha + 1)\,\partial_\tau + \alpha^2\, x\,\partial_x + y\,\partial_y + (\alpha + 1)^2\, z\,\partial_z. \tag{3.17}$$

The above metric was first given, although in a different form, by Kasner [19]. The metric is particularly interesting for the values $(\alpha, \beta) = (1, -1/2)$ or $(\alpha, \beta) = (-1/2, 1)$: In addition to the three Killing fields (3.3), there is a fourth of the form

$$\xi_4 = y\,\partial_x - x\,\partial_y. \tag{3.18}$$

The non permitted pair of values $(\alpha, \beta) = (0, 0)$ leads the metric (3.16) to the standard Minkowski form, so we can include these values into the domain of the constants $(\alpha, \beta)$.

## 3.2 One real and two complex conjugate eigenvalues

In this case, the matrix $\vartheta$ cannot be diagonalized, but we can choose an apprpriate $\Lambda$ such as to bring $\vartheta$ into its "rational normal form", i.e.,

$$\widetilde{\vartheta} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -(p_2^2 + p_3^2) & 2\,p_2 \end{pmatrix} \tag{3.19}$$

with eigenvalues $(p_1, p_2+\mathrm{i}\,p_3, p_2-\mathrm{i}\,p_3)$. Thus, we have for the exponential matrix $\exp(\widetilde{\vartheta}\,t)$

$$\begin{pmatrix} e^{p_1\,t} & 0 & 0 \\ 0 & \dfrac{e^{t\,p_2}}{p_3}\,(-p_2\,\sin(p_3\,t)+p_3\,\cos(p_3\,t)) & \dfrac{e^{p_2\,t}}{p_3}\,\sin p_3\,t \\ 0 & -\dfrac{e^{p_2\,t}}{p_3}\,\left(p_2^2+p_3^2\right)\,\sin(p_3\,t) & \dfrac{e^{t\,p_2}}{p_3}\,(p_2\,\sin(p_3\,t)+p_3\,\cos(p_3\,t)) \end{pmatrix} \tag{3.20}$$

and the constants $(p_1, p_2, p_3)$ are related through (3.8):

$$2\,p_1\,p_2+p_2^2+p_3^2=0 \tag{3.21}$$

Since the constants $p_1$, $p_3$ cannot vanish, we can divide the previous relation by $p_1^2\,p_3$ obtaining

$$2\,\lambda+\beta\,\lambda^2+\beta=0 \Rightarrow \beta=-\frac{2\,\lambda}{\lambda^2+1}, \quad \beta=\frac{p_3}{p_1},\ \lambda=\frac{p_2}{p_3} \tag{3.22}$$

Utilizing the fact that the matrix $\gamma_{\alpha\beta}$ must be symmetric as well as the freedom of using the automorphisms, we get, after the transformation $\tau=\beta\,p_1\,t$, the following metric:

$$\begin{aligned} d\,s^2 &= e^{(2\,\lambda+\beta^{-1})\,\tau}\,d\,\tau^2+e^{\tau/\beta}\,d\,x^2-e^{\lambda\,\tau}\,\sin\tau\,d\,y^2+ \\ &+2\,e^{\lambda\,\tau}\,\cos\tau\,d\,y\,d\,z+e^{\lambda\,\tau}\,\sin\tau\,d\,z^2 \end{aligned} \tag{3.23}$$

which possesses a homothety produced by the field

$$H=-4\lambda\,\partial_\tau-4\,\lambda^2\,x\,\partial_x+\big(y\,(-\lambda^2+1)+2\,\lambda\,z\big)\,\partial_y-\big(z\,(\lambda^2-1)+2\,\lambda\,y\big)\,\partial_z \tag{3.24}$$

This metric was first given, although produced in a different way, by Harrison [20]. Also in this case there are special values of the constant $\lambda$ for which we have a fourth Killing field:

$$\lambda=\frac{\sqrt{3}}{3} \Rightarrow \xi_4=6\,\partial_\tau+2\,\sqrt{3}\,x\,\partial_x-(\sqrt{3}\,y+3\,z)\,\partial_y+(3\,y-\sqrt{3}\,z)\,\partial_z$$

$$\lambda=-\frac{\sqrt{3}}{3} \Rightarrow \xi_4=6\,\partial_\tau-2\,\sqrt{3}\,x\,\partial_x+(\sqrt{3}\,y-3\,z)\,\partial_y+(3\,y+\sqrt{3}\,z)\,\partial_z,$$

while there is **no** homothety. Finally, it is worth noting that in metric (3.23) the hypersurface $t=$ const. is spacelike.

## 3.3 Three real (two repeated) eigenvalues

In this case, the matrix $\vartheta$ can be brought in its Jordan normal form by a proper choice of the matrix $\Lambda$, i.e.

$$\widetilde{\vartheta}=\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 1 \\ 0 & 0 & p_2 \end{pmatrix} \Rightarrow \exp(\widetilde{\vartheta}\,t)=\begin{pmatrix} e^{p_1\,t} & 0 & 0 \\ 0 & e^{p_2\,t} & e^{p_2\,t}\,t \\ 0 & 0 & e^{p_2\,t} \end{pmatrix} \tag{3.25}$$

where the constants $(p_1, p_2)$ are related through (3.8):

$$p_2\,(p_2 + 2\,p_1) = 0 \tag{3.26}$$

Here too, the constant $p_1$ is not zero, so dividing the previous relation by $p_1^2$ yields

$$\lambda\,(\lambda+2) = 0 \Rightarrow \lambda = 0 \,\vee\, \lambda = -2, \quad \frac{p_2}{p_1} = \lambda \tag{3.27}$$

With the help of the automorphisms and the transformation $\tau = p_1\,t$ we get the metric

$$d\,s^2 = e^{(2\,\lambda+1)\,\tau}\,d\,\tau^2 + e^{\tau}\,d\,x^2 + \tau\,e^{\lambda\,\tau}\,d\,y^2 + 2\,e^{\lambda\,\tau}\,d\,y\,d\,z \tag{3.28}$$

This metric is also a member of the Harrison class [20] and admits a homothety produced by the field

$$H = 2\,\partial_\tau + 2\,x\,\lambda\,\partial_x + (\lambda+1)\,y\,\partial_y + (z\,(\lambda+1) - y)\,\partial_z \tag{3.29}$$

In this case too, the hypersurface $t = \text{const.}$ is spacelike. Furthermore, for the value $\lambda = 0$ the metric (3.28) describes a pp-wave, since the Killing field $u = \xi_3 = \partial_z$ has zero covariant derivative and zero measure:

$$\lambda = 0 \Rightarrow u^\alpha\,u_\alpha = 0 \,\wedge\, u^\alpha{}_{;\beta} = 0 \tag{3.30}$$

## 3.4 Three real repeated eigenvalues

In this case, the Jordan normal form of the matrix $\vartheta$ is

$$\widetilde{\vartheta} = \begin{pmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{pmatrix} \Rightarrow \exp(\widetilde{\vartheta}\,t) = e^{p\,t} \begin{pmatrix} 1 & t & \frac{1}{2}\,t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.31}$$

so (3.8) gives $p = 0$. Again, exploiting the fact that the matrix $\gamma_{\alpha\beta}$ is symmetric as well the automorphism matrices, we get the metric

$$d\,s^2 = d\,t^2 + 2\,t^2\,d\,x^2 + d\,y^2 - 4\,t\,d\,x\,d\,y + 4\,d\,x\,d\,z \tag{3.32}$$

which has the following properties:

- There is a homothety, with corresponding field

$$H = t\,\partial_t + y\,\partial_y + 2\,z\,\partial_z \tag{3.33}$$

- It is a pp-wave metric since, for the Killing field $u = \xi_3 = \partial_z$, we have

$$u^\alpha\,u_\alpha = 0 \,\wedge\, u^\alpha{}_{;\beta} = 0 \tag{3.34}$$

- The hypersurface $t = \text{const.}$ is spacelike.

At this point, the question arises whether the two metrics (3.28) with $\lambda = 0$ and (3.32) are actually the same since they both are pp-wave metrics, they have three Killing fields one of which is timelike, and they both admit a homothety. The classical way to answer this question is by turning to the scalar invariants that can be constructed by the Riemann tensor and its covariant derivatives. This approach is not applicable to our case, since the fact that the metrics are pp-wave metrics implies the vanishing of all these scalar invariants. An alternative approach is to find a tensor that is identically zero for one of the metrics and non zero for the other. This would mean that there is no coordinate transformation relating the two metrics. If we consider the tensor

$$\Pi_{\alpha\beta\gamma\delta\epsilon} = R^{\kappa\ \ \lambda}_{\ \alpha\ \beta}\, R_{\kappa\gamma\delta\epsilon;\lambda}, \tag{3.35}$$

we observe that it vanishes identically for metric (3.32) but not for metric (3.28) with $\lambda = 0$. Thus, the two metrics describe different geometries.

# 4 Bianchi Type II

In this case, the structure constants are

$$\begin{aligned} C^1{}_{23} = -C^1{}_{32} = \frac{1}{2} & \\ C^\alpha{}_{\beta\gamma} = 0 \quad & \text{for all the other values of}\, \alpha,\ \beta,\ \gamma \end{aligned} \tag{4.1}$$

Utilizing these values and the definition (2.2) of the 1-forms $\boldsymbol{\sigma}^\alpha$, we get:

$$\sigma^\alpha{}_i = \begin{pmatrix} z & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\sigma}^1 = z\,\boldsymbol{d}\,x + \boldsymbol{d}\,y, \quad \boldsymbol{\sigma}^2 = \boldsymbol{d}\,z, \quad \boldsymbol{\sigma}^3 = \boldsymbol{d}\,x \tag{4.2}$$

with the corresponding Killing fields

$$\xi_1 = \partial_x, \quad \xi_2 = -x\,\partial_y + \partial_z, \quad \xi_3 = \partial_y \tag{4.3}$$

From (2.9) we have for the automorphisms $\Lambda^\alpha{}_\beta$ and the triplett $P^\alpha(t)$

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\, s_4 & x(t) & y(t) \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix} \tag{4.4a}$$

$$P^\alpha(t) = \left( P(t), \frac{e^{s_1}\,\dot{y} - s_3\,\dot{x}}{e^{s_1+s_2} - s_3\, s_4}, \frac{s_4\,\dot{y} - e^{s_2}\,\dot{x}}{e^{s_1+s_2} - s_3\, s_4} \right) \tag{4.4b}$$

where $P(t)$, $x(t)$ and $y(t)$ are the three aforementioned arbitrary functions of time which appear in the vector $P^\alpha$ and can be thus used to set the shift vector $N^\alpha$ to zero. The remaining symmetry of the field equations is described by the constant matrix

$$M^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\,s_4 & s_5 & s_6 \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix} \tag{4.5}$$

This matrix defines a transformation of the variables $\gamma_{\alpha\beta}$, $\tilde{\gamma}_{\alpha\beta} = M^\mu{}_\alpha M^\nu{}_\beta \gamma_{\mu\nu}$, which can be written analytically as

$$\begin{cases} \tilde{\gamma}_{11} = (e^{s_1+s_2} - s_3\,s_4)^2\,\gamma_{11} \\ \tilde{\gamma}_{12} = (e^{s_1+s_2} - s_3\,s_4)(e^{s_1}\,\gamma_{12} + s_4\,\gamma_{13} + s_5\,\gamma_{11}) \\ \tilde{\gamma}_{13} = (e^{s_1+s_2} - s_3\,s_4)(e^{s_2}\,\gamma_{13} + s_3\,\gamma_{12} + s_6\,\gamma_{11}) \\ \tilde{\gamma}_{22} = e^{2\,s_1}\,\gamma_{22} + 2\,e^{s_1}\,s_4\,\gamma_{23} + 2\,e^{s_1}\,s_5\,\gamma_{12} + s_4^2\,\gamma_{33} + 2\,s_4\,s_5\gamma_{13} + s_5^2\,\gamma_{11} \\ \tilde{\gamma}_{23} = s_3\,(e^{s_1}\,\gamma_{22} + s_4\,\gamma_{23} + s_5\,\gamma_{12}) + e^{s_2}\,(e^{s_1}\,\gamma_{23} + s_4\,\gamma_{33} + s_5\,\gamma_{13}) + \\ \qquad s_6\,(e^{s_1}\,\gamma_{12} + s_4\,\gamma_{13} + s_5\,\gamma_{11}) \\ \tilde{\gamma}_{33} = e^{2\,s_2}\,\gamma_{33} + s_3^2\,\gamma_{22} + 2\,s_3\,s_6\,\gamma_{12} + s_6^2\,\gamma_{11} + 2\,e^{s_2}\,(s_3\,\gamma_{23} + s_6\,\gamma_{13}) \end{cases} \tag{4.6}$$

These equations define a transformation group $G_r$ of dimension $r = dim(Aut(II)) = 6$. The six group generators are determined through the definition

$$X_A \equiv \left(\frac{\partial\tilde{\gamma}_{\alpha\beta}}{\partial s_A}\right)_{s=0} \frac{\partial}{\partial\gamma_{\alpha\beta}} \tag{4.7}$$

where $A = \{1, 2, 3, 4, 5, 6\}$. Application of this relation to (4.6) results in the following explicit form of the six generators:

$$\begin{aligned} X_1 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + 2\,\gamma_{12}\,\partial_{\gamma_{12}} + \gamma_{13}\,\partial_{\gamma_{13}} + 2\,\gamma_{22}\,\partial_{\gamma_{22}} + \gamma_{23}\,\partial_{\gamma_{23}} \\ X_2 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + \gamma_{12}\,\partial_{\gamma_{12}} + 2\,\gamma_{13}\,\partial_{\gamma_{13}} + \gamma_{23}\,\partial_{\gamma_{23}} + 2\,\gamma_{33}\,\partial_{\gamma_{33}} \\ X_3 &= \gamma_{12}\,\partial_{\gamma_{13}} + \gamma_{22}\,\partial_{\gamma_{23}} + 2\,\gamma_{23}\,\partial_{\gamma_{33}} \\ X_4 &= \gamma_{13}\,\partial_{\gamma_{12}} + 2\,\gamma_{23}\,\partial_{\gamma_{22}} + \gamma_{33}\,\partial_{\gamma_{23}} \\ X_5 &= \gamma_{11}\,\partial_{\gamma_{12}} + 2\,\gamma_{12}\,\partial_{\gamma_{22}} + \gamma_{13}\,\partial_{\gamma_{23}} \\ X_6 &= \gamma_{11}\,\partial_{\gamma_{13}} + \gamma_{12}\,\partial_{\gamma_{23}} + 2\,\gamma_{13}\,\partial_{\gamma_{33}} \end{aligned} \tag{4.8}$$

The algebra $g_r$ associated with the group $G_r$ has the following table of non-vanishing commutators:

$$\begin{array}{lll} [X_1, X_3] = X_3 & [X_1, X_4] = -X_4 & [X_1, X_6] = X_6 \\ [X_2, X_3] = -X_3 & [X_2, X_4] = X_4 & [X_2, X_5] = X_5 \\ [X_3, X_4] = X_1 - X_2 & [X_3, X_5] = -X_6 & [X_4, X_6] = -X_5 \end{array} \tag{4.9}$$

As it is implied by the previous commutators (4.9), the group is non abelian, hence we cannot simultaneously diagonalize all the generators. However, the generators $X_5, X_6$

commute with each other, as well as with the generator $Y_2$, so we can bring them into their normal form by the following transformation of the $\gamma_{\alpha\beta}$:

$$\begin{cases} \gamma_{11} = e^{u_1(t)},\ \gamma_{12} = e^{u_1 t}\, u_2(t),\ \gamma_{13} = e^{u_1(t)}\, u_3(t) \\ \gamma_{22} = e^{u_1(t)}\, (u_2^2(t) + u_4(t)) \\ \gamma_{23} = e^{u_1(t)}\, (u_2(t)\, u_3(t) + u_5(t)),\ \gamma_{33} = e^{u_1(t)}\, (u_3^2(t) + u_6(t)) \end{cases} \tag{4.10}$$

▶ **The solution space in the gauge $N^\alpha = 0$**

Before we proceed to the solution of the Einstein equations, we have to find the values allowed for the functions $u_i, i = 1, \ldots, 6$. The determinant of the matrix $\gamma_{\alpha\beta}$, is

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1}\,\left(u_4\,u_6 - u_5^2\right), \tag{4.11}$$

therefore $u_4\,u_6 - u_5^2 > 0$.

Starting from the equations $E_2 = 0$, $E_3 = 0$, we get the following system for the variables $\dot{u}_2$, $\dot{u}_3$:

$$E_2 = 0 \Rightarrow u_5\,\dot{u}_2 - u_4\,\dot{u}_3 = 0 \tag{4.12a}$$

$$E_3 = 0 \Rightarrow u_6\,\dot{u}_2 - u_5\,\dot{u}_3 = 0. \tag{4.12b}$$

This system admits non zero solutions only if $u_4\,u_6 - u_5^2 = 0$, a condition forbidden by our restrictions. Thus, we have

$$u_2(t) = k_2, \quad u_3(t) = k_3 \tag{4.13}$$

Now, by using the automorphism matrix

$$M^\alpha{}_\beta = \begin{pmatrix} 1 & -k_2 & -k_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.14}$$

the matrix $\gamma_{\alpha\beta}$ is brought to block diagonal form, and we have the following theorem: *The general solution of Bianchi Type II corresponds to a block diagonal form in the basis of the 1-forms $\boldsymbol{\sigma}^\alpha$.*

As we have now proved that the matrix $\gamma_{\alpha\beta}$ is in block diagonal form with no loss of generality, we can repeat the former procedure for the calculation of the generators, i.e., we can start from the matrix

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{23} & \gamma_{33} \end{pmatrix} \tag{4.15}$$

with the corresponding automorphism matrix

$$M^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\,s_4 & 0 & 0 \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix}. \tag{4.16}$$

The generators thus obtained are

$$
\begin{aligned}
X_1 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + 2\,\gamma_{22}\,\partial_{\gamma_{22}} + \gamma_{23}\,\partial_{\gamma_{23}} \\
X_2 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + \gamma_{23}\,\partial_{\gamma_{23}} + 2\,\gamma_{33}\,\partial_{\gamma_{33}} \\
X_3 &= \gamma_{22}\,\partial_{\gamma_{23}} + 2\,\gamma_{23}\,\partial_{\gamma_{33}} \\
X_4 &= 2\,\gamma_{23}\,\partial_{\gamma_{22}} + \gamma_{33}\,\partial_{\gamma_{23}}.
\end{aligned}
\tag{4.17}
$$

The generators $Y_2$, $X_1 + X_2$, $X_3$ commute with each other, so they can be brought to their normal form by applying the transformation

$$
\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+4\,u_6} & 0 & 0 \\ 0 & e^{u_1+2\,u_6}\,u_4 & e^{u_1+2\,u_6}\,u_4\,u_5 \\ 0 & e^{u_1+2\,u_6}\,u_4\,u_5 & e^{u_1+2\,u_6}(u_4\,u_5^2+1) \end{pmatrix}. \tag{4.18}
$$

In this parameterization, the only variable of which the second derivative appears in the field equations is $u_4$, since the generators have been transformed to

$$
Y_2 = \frac{\partial}{\partial u_1},\; X_3 = \frac{\partial}{\partial u_5},\; X_1 + X_2 = \frac{\partial}{\partial u_6} \tag{4.19}
$$

To further simplify the present situation, we shall also use the generator $Y_1 = \frac{\partial}{\partial t}$. The variable $u_4$, that does not appear in the generators $Y_2$, $X_1 + X_2$, $X_3$, can be utilized in order to parameterize time $t$, i.e., we can apply the transformation

$$
t \to u_4(t) = s,\; u_1(t) \to u_1(s),\; u_5(t) \to u_5(s),\; u_6(t) \to u_6(s) \tag{4.20}
$$

Of course, this transformation can be applied only in the case where $u_4(t)$ is not a constant function. hence, we must distinguish between two cases: either the function $u_4(t)$ is constant or not.

## 4.1 The case $u_4(t) = $ const.

Let $u_4(t) = k_4$. Then, the linear constraints (2.4) are identically satisfied, while the quadratic constraint (2.3) defines the lapse function $N^2(t)$ as

$$
N^2 = k_4\,e^{u_1}\left(3\,\dot{u}_1^2 - k_4\,\dot{u}_5^2 + 16\,\dot{u}_1\,\dot{u}_6 + 20\,\dot{u}_6^2\right) \tag{4.21}
$$

Substituting the latter into the dynamic equations (2.5), we can solve them for the variables $\ddot{u}_1$, $\ddot{u}_5$, $\ddot{u}_6$. Starting from equation $E_{11} = 0$, we notice that the coefficient of $\ddot{u}_5$ is proportional to the quantity

$$
\dot{u}_5\,(\dot{u}_1 + 4\,\dot{u}_6). \tag{4.22}
$$

Therefore, in order to solve for $\ddot{u}_5$ we must be sure that this coefficient is not zero. The case $\dot{u}_1 + 4\,\dot{u}_6 = 0$ leads to a zero lapse, so only the case $u_5(t) = k_5$ is left.

Assuming that $u_5(t) \neq k_5$, we solve equation $E_{11} = 0$ for $\ddot{u}_5$ and, substituting in equation $E_{22} = 0$, we can solve the latter for $\ddot{u}_1$. The coefficient of $\ddot{u}_1$ is the quantity $k_4\,\dot{u}_6$ which cannot vanish since, for $k_4 = 0$, we have $det(\gamma_{\alpha\beta}) = 0$ while, for $u_6(t) = k_6$, we obtain a zero lapse. Now, substituting in $E_{23} = 0$ we get

$$k_4\,\dot{u}_5\,(\dot{u}_1 + 3\,\dot{u}_6) = 0 \Rightarrow u_1 = k_1 - 3\,u_6. \tag{4.23}$$

However, this leads to $u_5 = k_5$, which is impossible in this "branch" of the solution.

If we start with $u_5 = k_5$, then the coefficient of $\ddot{u}_1$ in equation $E_{11} = 0$ is proportional to the quantity

$$\dot{u}_6\,(\dot{u}_1 + 3\,\dot{u}_6) \tag{4.24}$$

The case $u_6 = k_6$ leads to a zero lapse, so the case remaining to be considered, before we solve equation $E_{11} = 0$ for the variable $\ddot{u}_1$ , is

$$u_6(t) = k_6 - \frac{u_1(t)}{3}. \tag{4.25}$$

With this value for the function $u_6(t)$ the rest of the equations are satisfied identically, but the lapse function takes now the value

$$N^2 = -\frac{1}{9}e^{u_1(t)}\,k_4\,\dot{u}_1^2, \tag{4.26}$$

i.e., we have ended up with a metric of **Euclidean** signature. Using the automorphism matrix

$$M = \begin{pmatrix} \sqrt{k_4}\,e^{-2\,k_6} & 0 & 0 \\ 0 & \sqrt{k_4\,k_5^2+1}\,e^{-k_6} & 0 \\ 0 & -\frac{k_4\,k_5}{\sqrt{k_4\,k_5^2+1}}\,e^{-k_6} & \frac{\sqrt{k_4}}{\sqrt{k_4\,k_5^2+1}}\,e^{-k_6} \end{pmatrix}, \tag{4.27}$$

the matrix $\gamma_{\alpha\beta}$ can be brought in diagonal form. By choosing a gauge in which $u_1(t) = 3\,\ln t$, we get the metric

$$d\,s^2 = t\,\boldsymbol{d}\,t^2 + \frac{1}{t}\,(\boldsymbol{\sigma}^1)^2 + t\,(\boldsymbol{\sigma}^2)^2 + t\,(\boldsymbol{\sigma}^3)^2 \tag{4.28}$$

In addition to the three Killing fields (4.3), this metric has a fourth one:

$$\xi_4 = -2\,z\,\partial_x + (z^2 - x^2)\,\partial_y + 2\,x\,\partial_z \tag{4.29}$$

and it admits a homothety produced by the field

$$H = t\,\partial_t + x\,\partial_x + 2\,y\,\partial_y + z\ \partial_z \tag{4.30}$$

Since we have examined the case in which the coefficient of $\ddot{u}_1$ in $E_{11} = 0$ vanishes, we can solve this equation for this function, so

$$\ddot{u}_1 = \frac{1}{\dot{u}_6}\left(3\,\dot{u}_1^3 + 25\,\dot{u}_1^2\,\dot{u}_6 + (68\,\dot{u}_6^2 + \ddot{u}_6)\,\dot{u}_1 + 60\,\dot{u}_6^3\right), \tag{4.31}$$

hence the rest of the equations are also satisfied. In order to simplify the latter equation we choose a gauge in which $u_6(t) = 3\,t$, thus obtaining the autonomous first order equation for $\dot{u}_1$

$$\ddot{u}_1 = \dot{u}_1^3 + 25\,\dot{u}_1^2 + 204\,\dot{u}_1 + 540 \tag{4.32}$$

This equation can be integrated with the help of the transformation

$$t = y(\xi), \quad \dot{u}_1 = \xi, \quad \ddot{u}_1 = \frac{1}{y'(\xi)} \tag{4.33}$$

which brings the equation in the following form:

$$\begin{aligned} y'(\xi) &= \frac{1}{(\xi+10)(\xi+9)(\xi+6)} \Rightarrow \\ y(\xi) &= k_1 + \frac{1}{4}\,\ln(\xi+10) - \frac{1}{3}\,\ln(\xi+9) + \frac{1}{12}\,\ln(\xi+6) \end{aligned} \tag{4.34}$$

Using the automorphism matrix

$$M = \begin{pmatrix} -\frac{\mu\,\sqrt[4]{3}}{\sqrt{2}}\,e^{-6\,k_1-k_2/2} & 0 & 0 \\ 0 & \frac{\sqrt[4]{27}}{2}\,e^{-3\,k_1} & \sqrt{\frac{2}{3}}\,\mu\,k_5\,e^{-3\,k_1-k_2/2} \\ 0 & 0 & -\sqrt{\frac{2}{3}}\,\mu\;e^{-3\,k_1-k_2/2} \end{pmatrix} \tag{4.35}$$

and redefining the constant $k_4$ and time $\xi$ as

$$k_4 = \frac{8\sqrt{3}}{27}\,\mu^2\,e^{-k_2}, \quad \xi = -6\,\frac{5\,e^{2\,\psi}-1}{3\,e^{2\,\psi}-1} \tag{4.36}$$

we obtain the metric

$$d\,s^2 = \mu^2\left(-e^{2\,\psi}\,\cosh\psi\,\boldsymbol{d}t^2 + \mathrm{sech}\psi\,(\boldsymbol{\sigma}^1)^2 + \;e^{\psi}\,\cosh\psi\,(\boldsymbol{\sigma}^2)^2 + e^{\psi}\,\cosh\psi\,(\boldsymbol{\sigma}^3)^2\right) \tag{4.37}$$

In addition to the three Killing fields (4.3), this metric has a fourth one:

$$\xi_4 = -2\,z\,\partial_x + (z^2 - x^2)\,\partial_y + 2\,x\,\partial_z \tag{4.38}$$

and it does **not** admit a homothety, so the constant $\mu$ cannot be absorbed.

## 4.2 The case $u_4(t) \neq$ const.

Let us assume that $u_4(t) = t$. Then the linear constraints (2.4) are satisfied identically, while the quadratic constraint (2.3) defines the following lapse function $N^2(t)$:

$$N^2 = e^{u_1}\left(3\,t\,\dot{u}_1^2 - t^2\,\dot{u}_5^2 + 2\,(8\,t\,\dot{u}_6 + 1)\dot{u}_1 + 2\,(10\,t\,\dot{u}_6 + 3)\,\dot{u}_6\right) \tag{4.39}$$

Substituting this in equation $E_{11} = 0$, we find that the coefficient of $\ddot{u}_5$ is proportional to the quantity $\dot{u}_5\,(\dot{u}_1 + 4\,\dot{u}_6)$ which cannot vanish since it leads to a zero lapse. Solving equation $E_{11} = 0$ for $\ddot{u}_5$ and then substituting in equation $E_{22} = 0$, we find that the coefficient of $\ddot{u}_6$ is proportional to the quantity $t\,\dot{u}_1 + 2$ which cannot vanish because in such a case we obtain again a zero lapse. Thus, we can solve for $\ddot{u}_6$ and substitute in the rest of the equations. Equation $E_{23} = 0$ includes the function $\ddot{u}_1$ the coefficient of which is proportional to the quantity

$$6\,t\,\dot{u}_6 + 2\,t\,\dot{u}_1 + 1, \tag{4.40}$$

the vanishing of which leads, as we shall see, to a metric of Euclidean signature. Assuming that the quantity(4.40) is not zero, we get the following system of differential equations:

$$\ddot{u}_5 = -\frac{2\,\dot{u}_5}{t}(t^3\,\dot{u}_5^2 + 1) \tag{4.41a}$$

$$\ddot{u}_6 = \frac{1}{2\,t}\left(-6\,t\,\dot{u}_1^2 + t^2\,\dot{u}_5^2(-4\,t\,\dot{u}_6 + 3) - 4\,\dot{u}_1(8\,t\,\dot{u}_6 + 1) - 2\,\dot{u}_6(20\,t\,\dot{u}_6 + 7)\right) \tag{4.41b}$$

$$\ddot{u}_1 = \frac{1}{t}\left(9\,t\,\dot{u}_1^2 - 5\,t^2\,\dot{u}_5^2 + 6\,\dot{u}_6(10\,t\,\dot{u}_6 + 3) + (48\,t\,\dot{u}_6 - 2\,t^3\,\dot{u}_5^2 + 5)\,\dot{u}_1\right) \tag{4.41c}$$

We shall first consider the case of a vanishing quantity (4.40):

$$6\,t\,\dot{u}_6 + 2\,t\,\dot{u}_1 + 1 = 0 \Rightarrow u_6 = k_6 - \frac{1}{6}\,\ln t - \frac{1}{3}\,u_1 \tag{4.42}$$

Solving equation $E_{11} = 0$ for $\ddot{u}_5$ and substituting in $E_{22} = 0$, we obtain

$$\dot{u}_5 = 0 \Rightarrow u_5 = k_5 \tag{4.43}$$

that brings equation $E_{33} = 0$ into the form

$$3\,t^2\,\ddot{u}_1 + t^2\,\dot{u}_1^2 + 7\,t\,\dot{u}_1 + 4 = 0. \tag{4.44}$$

The latter is a Riccati equation for $w_1(t) = \dot{u}_1(t)$

$$\dot{w}_1 = -\frac{1}{3}\,w_1^2 - \frac{7}{3\,t}\,w_1 - \frac{4}{3\,t^2}, \tag{4.45}$$

which is easily solvable since a partial solution of it is already known: $w_1 = -\frac{2}{t}$. With the transformation

$$w_1(t) = -\frac{2}{t} + \frac{1}{h(t)}, \tag{4.46}$$

the equation (4.45) takes the linear form

$$\dot{h} = \frac{1}{t}\,h + \frac{1}{3} \Rightarrow h(t) = \frac{t}{3}\,\ln t + k_1\,t. \tag{4.47}$$

By integrating, we finally have for $u_1(t)$

$$u_1 = k_2 - 2\,\ln t + 3\,\ln(\ln t + 3\,k_1). \tag{4.48}$$

Redefining the constants $k_2$, $k_6$

$$k_2 = \ln \mu^2 - 3\,k_1, \qquad k_6 = \frac{1}{24}\,(3\,\ln \kappa^2 - \ln \mu^2 + 3\,k_1), \tag{4.49}$$

using the automorphism matrix

$$M = \begin{pmatrix} \sqrt{\frac{\mu^3}{\kappa}}\,e^{-3\,k_1/4} & 0 & 0 \\ 0 & \sqrt[4]{\frac{\mu^3}{\kappa}}\,e^{3\,k_1/8} & -\sqrt[4]{\frac{\mu^3}{\kappa}}\,k_5\,e^{-9\,k_1/8} \\ 0 & 0 & \sqrt[4]{\frac{\mu^3}{\kappa}}\,e^{-9\,k_1/8} \end{pmatrix}, \tag{4.50}$$

and putting $t = e^{\xi - 3\,k_1}$, we obtain the metric of **Euclidean** signature

$$d\,s^2 = \mu^2 \left( e^{-\xi}\,\xi\,\boldsymbol{d}\,\xi^2 + \frac{1}{\xi}\,(\boldsymbol{\sigma}^1)^2 + \xi\,(\boldsymbol{\sigma}^2)^2 + e^{-\xi}\,\xi\,(\boldsymbol{\sigma}^3)^2 \right). \tag{4.51}$$

This metric does **not** admit a homothety, so the constant $\mu$ cannot be absorbed.

Now, returning to the system (4.41) and multiplying the first equation by $\dot u_5$, we have

$$\begin{aligned} \dot u_5\,\ddot u_5 &= -\frac{2\,\dot u_5^2}{t}(t^3\,\dot u_5^2 + 1) \Rightarrow \frac{d\,\dot u_5^2}{d\,t} = -\frac{4\,\dot u_5^2}{t}(t^3\,\dot u_5^2 + 1) \Rightarrow \\ \dot y &= -\frac{4\,y}{t}(t^3\,y + 1), \quad y = \dot u_5^2, \end{aligned} \tag{4.52}$$

i.e. a Ricatti equation for $y(t)$. A partial solution of this equation is $y_1 = -\frac{1}{4\,t^3}$, hence we can reduce it to a linear differential equation by applying the transformation

$$y = -\frac{1}{4\,t^3} + \frac{1}{h} \tag{4.53}$$

and get

$$\dot h = \frac{2}{t}\,h + 4\,t^2 \Rightarrow h = -\frac{4}{k_{51}}\,t^2 + 4\,t^3 \Rightarrow u_5 = k_{52} \pm \sqrt{k_{51} - \frac{1}{t}}. \tag{4.54}$$

Consequently, the last two equations of the system (4.41) become

$$\ddot u_1 = \langle \dot u_1|\,A_1\,|\dot u_6\rangle, \quad \ddot u_6 = \langle \dot u_1|\,A_2\,|\dot u_6\rangle, \tag{4.55}$$

where use has been made of the notation $\langle \dot u_i| = (1\,\dot u_i\,\dot u_i^2)$ and $|\dot u_i\rangle = \langle \dot u_i|^t$ with the $3 \times 3$ matrices $A_1$, $A_2$ given by the relations

$$A_1 = \begin{pmatrix} -\dfrac{5}{4\,t^2\,(k_{51}\,t - 1)} & \dfrac{18}{t} & 60 \\ \dfrac{10\,k_{51}\,t - 11}{2\,k_{51}\,t^2 - 2\,t} & 48 & 0 \\ 9 & 0 & 0 \end{pmatrix} \tag{4.56}$$

$$A_2 = \begin{pmatrix} \dfrac{3}{8\,t^2\,(k_{51}\,t-1)} & -\dfrac{14\,k_{51}\,t-13}{2\,k_{51}\,t^2-2\,t} & -20 \\ -\dfrac{2}{t} & -16 & 0 \\ -3 & 0 & 0 \end{pmatrix}. \tag{4.57}$$

The latter system is of polynomial form in the variables $\dot{u}_1$, $\dot{u}_6$, hence we can simplify it based on the transformation

$$\dot{u}_1 = \frac{1}{2\,\sqrt{t\,(k_{51}\,t-1)}}\,(y_1+5\,y_2) - \frac{1}{t} \tag{4.58a}$$

$$\dot{u}_6 = -\frac{1}{4\,\sqrt{t\,(k_{51}\,t-1)}}\,(y_1+3\,y_2) + \frac{1}{4\,t}. \tag{4.58b}$$

Thus, we obtain the system

$$\dot{y}_1 = -\frac{4\,y_1\,y_2+k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}}, \qquad \dot{y}_2 = -\frac{4\,y_1\,y_2+k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}} \tag{4.59}$$

i.e.,

$$\dot{y}_2 = \dot{y}_1 \Rightarrow y_2 = y_1 + k_2\,\sqrt{k_{51}}, \qquad \dot{y}_1 = -\frac{4\,y_1^2+4\,k_2\,\sqrt{k_{51}}\,y_1+k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}}. \tag{4.60}$$

The previous Riccati equation is simplified by applying the transformation

$$t \mapsto \frac{1}{k_{51}}\,\cosh^2(2\,\tau),\ y_1(t) \mapsto \sqrt{k_{51}}\,y_1(\tau),\ k_2 \mapsto -\cosh\mu, \tag{4.61}$$

thus arriving at the equation

$$\begin{gathered} \dot{y}_1 + 4\,y_1^2 - 4\,\cosh\mu\,y_1 + 1 = 0 \Rightarrow \\ y_1 = \frac{1}{2}\left[\cosh\mu - \sinh\mu\,\tanh\big(2\,(k_1-\tau)\,\sinh\mu\big)\right]. \end{gathered} \tag{4.62}$$

Substituting $y_1(\tau)$ in (4.58), we obtain

$$u_1 = c_1 - 4\,\tau\,\cosh\mu + 3\,\ln(\cosh(2\,(k_1-\tau)\,\sinh\mu)) + \ln\frac{\tanh 2\,\tau}{\sinh 4\,\tau} \tag{4.63a}$$

$$u_6 = c_2 + \tau\,\cosh\mu - \ln(\cosh(2\,(k_1-\tau)\,\sinh\mu)) - \frac{1}{4}\ln\frac{\tanh 2\,\tau}{\sinh 4\,\tau} \tag{4.63b}$$

In order to get the final form of the metric, we redefine the constants $\mu = 2\,\sigma$, $k_{51} = \exp(c_1 - 4\,k_1\,\cosh 2\,\sigma)/2\,\kappa^2$, we choose the gauge $\tau = k_1 - \xi\,\mathrm{csch} 2\,\sigma/2$ and we use the automorphism matrices

$$M = \begin{pmatrix} -\kappa\,e^{-\frac{c_1}{2}-2\,c_2} & 0 & 0 \\ 0 & 0 & \sqrt[4]{2}\,e^{-c_2+k_1-k_1\,\cosh 2\,\sigma} \\ 0 & \frac{\kappa}{\sqrt[4]{2}}\,e^{-\frac{c_1}{2}-c_2-k_1+k_1\,\cosh 2\,\sigma} & -\frac{\kappa}{\sqrt[4]{2}}\,e^{-\frac{c_1}{2}-c_2+k_1+k_1\,\cosh 2\,\sigma} \end{pmatrix}$$

for $k_{52} = \sqrt{k_{51}}$ and

$$M = \begin{pmatrix} d_1\, d_2 - d_3\, d_4 & 0 & 0 \\ 0 & d_1 & d_3 \\ 0 & d_4 & d_2 \end{pmatrix}$$

with

$$d_1 = -\frac{e^{\frac{c_1}{2}} + \sqrt{2}\,\kappa\, k_{52}\, e^{2\,k_1\,\cosh 2\,\sigma}}{\sqrt[4]{8}}\, e^{-\frac{c_1}{2} - c_2 + k_1 - k_1\,\cosh 2\,\sigma} \tag{4.64}$$

$$d_2 = \frac{\kappa}{\sqrt[4]{2}}\, e^{-\frac{c_1}{2} - c_2 - k_1 + k_1\,\cosh 2\,\sigma} \tag{4.65}$$

$$d_3 = \frac{e^{\frac{c_1}{2}} - \sqrt{2}\,\kappa\, k_{52}\, e^{2\,k_1\,\cosh 2\,\sigma}}{\sqrt[4]{8}}\, e^{-\frac{c_1}{2} - c_2 - k_1 - k_1\,\cosh 2\,\sigma} \tag{4.66}$$

$$d_4 = \frac{\kappa}{\sqrt[4]{2}}\, e^{-\frac{c_1}{2} - c_2 + k_1 + k_1\,\cosh 2\,\sigma} \tag{4.67}$$

for $k_{52} \neq \sqrt{k_{51}}$. Thus, we arrive at the form

$$d\,s^2 = \kappa^2 \left( e^{2\,\xi\,\coth 2\,\sigma}\,\cosh\xi\,\boldsymbol{d}\,\xi^2 + \mathrm{sech}\xi\,(\boldsymbol{\sigma}^1)^2 + e^{\xi\,\coth\sigma}\,\cosh\xi\,(\boldsymbol{\sigma}^2)^2 + e^{\xi\,\tanh\sigma}\,\cosh\xi\,(\boldsymbol{\sigma}^3)^2 \right) \tag{4.68}$$

This solution was first produced by Taub [21]. It does not admit a homothety, hence the constant $\kappa$ cannot be absorbed. It is noteworthy, that in the limiting case $\sigma \to +\infty$, this metric reduces to metric (4.37) which possesses four Killing fields.

# 5 Bianchi Type V

In this case, the structure constants are:

$$\begin{aligned} &C^1{}_{13} = -C^1{}_{31} = C^2{}_{23} = -C^2{}_{32} = \frac{1}{2} \\ &C^\alpha{}_{\beta\gamma} = 0 \qquad\qquad \text{for all the other values of } \alpha,\, \beta,\, \gamma \end{aligned} \tag{5.1}$$

Using these values in the definition (4.7) of the 1-forms $\boldsymbol{\sigma}^\alpha$ we have

$$\sigma^\alpha{}_i = \begin{pmatrix} 0 & 0 & e^{-x} \\ 0 & e^{-x} & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \boldsymbol{\sigma}^1 = e^{-x}\,\boldsymbol{d}\,z,\; \boldsymbol{\sigma}^2 = e^{-x}\,\boldsymbol{d}\,y,\; \boldsymbol{\sigma}^3 = \boldsymbol{d}\,x \tag{5.2}$$

with corresponding Killing fields

$$\xi_1 = \partial_z, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_x + y\,\partial_y + z\,\partial_z \tag{5.3}$$

The time-dependent automorphisms are:

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \rho_1\, P(t) & \rho_2\, P(t) & x(t) \\ \rho_3\, P(t) & \rho_4\, P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_1\,\rho_4 - \rho_2\,\rho_3 = 1 \tag{5.4}$$

and

$$P^{\alpha}(t) = \left(x(t)\left(\ln\frac{x(t)}{P(t)}\right)', y(t)\left(\ln\frac{y(t)}{P(t)}\right)', -\left(\ln P(t)\right)'\right) \tag{5.5}$$

where $P(t)$, $x(t)$ and $y(t)$ are arbitrary functions of time that can be utilized in order to set the shift vector to zero. The remaining symmetry of the field equations is described by the constant matrix:

$$M = \begin{pmatrix} e^{s_1} & s_2 & s_3 \\ s_4 & e^{s_5} & s_6 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

where we have changed the parameterization of the constants so that the choice $s_i = 0$ corresponds to the identity matrix. The transformation of the variables $\gamma_{\alpha\beta}$ defined by the matrix above is the $\tilde{\gamma}_{\alpha\beta} = M^{\mu}{}_{\alpha} M^{\nu}{}_{\beta} \gamma_{\mu\nu}$. This transformation reads analytically:

$$\begin{cases} \tilde{\gamma}_{11} = e^{2\,s_1}\,\gamma_{11} + s_4\,\left(2\,e^{s_1}\,\gamma_{12} + s_4\,\gamma_{22}\right) \\ \tilde{\gamma}_{12} = e^{s_1}\,s_2\,\gamma_{11} + \left(e^{s_1+s_5} + s_2\,s_4\right)\,\gamma_{12} + e^{s_5}\,s_4\,\gamma_{22} \\ \tilde{\gamma}_{13} = e^{s_1}\,s_3\,\gamma_{11} + \left(e^{s_1}\,s_6 + s_3\,s_4\right)\,\gamma_{12} + e^{s_1}\,\gamma_{13} + s_4\,s_6\,\gamma_{22} + s_4\,\gamma_{23} \\ \tilde{\gamma}_{22} = e^{2\,s_5}\,\gamma_{22} + 2\,e^{s_5}\,s_2\,\gamma_{12} + s_2^2\,\gamma_{11} \\ \tilde{\gamma}_{23} = e^{s_5}\,\gamma_{23} + e^{s_5}\,s_6\,\gamma_{22} + s_2\,\gamma_{13} + \left(e^{s_5}\,s_3 + s_2\,s_6\right)\,\gamma_{12} + s_2\,s_3\,\gamma_{11} \\ \tilde{\gamma}_{33} = \gamma_{33} + 2\,s_6\,\gamma_{23} + s_6^2\,\gamma_{22} + 2\,s_3\,\gamma_{13} + 2\,s_3\,s_6\,\gamma_{12} + s_3^2\,\gamma_{11} \end{cases} \tag{5.7}$$

The aforementioned equations define a transformation group $G_r$ of dimension $r = dim(Aut(IV)) = 6$. In exact analogy to the previous Type II case (see (4.7)) the six group generators are found to be:

$$\begin{aligned} X_1 &= 2\,\gamma_{11}\frac{\partial}{\partial\gamma_{11}} + \gamma_{12}\frac{\partial}{\partial\gamma_{12}} + \gamma_{13}\frac{\partial}{\partial\gamma_{13}} \\ X_2 &= \gamma_{11}\frac{\partial}{\partial\gamma_{12}} + 2\,\gamma_{12}\frac{\partial}{\partial\gamma_{22}} + \gamma_{13}\frac{\partial}{\partial\gamma_{23}} \\ X_3 &= \gamma_{11}\frac{\partial}{\partial\gamma_{13}} + \gamma_{12}\frac{\partial}{\partial\gamma_{23}} + 2\,\gamma_{13}\frac{\partial}{\partial\gamma_{33}} \\ X_4 &= 2\,\gamma_{12}\frac{\partial}{\partial\gamma_{11}} + \gamma_{22}\frac{\partial}{\partial\gamma_{12}} + \gamma_{23}\frac{\partial}{\partial\gamma_{13}} \\ X_5 &= \gamma_{12}\frac{\partial}{\partial\gamma_{12}} + 2\,\gamma_{22}\frac{\partial}{\partial\gamma_{22}} + \gamma_{23}\frac{\partial}{\partial\gamma_{23}} \\ X_6 &= \gamma_{12}\frac{\partial}{\partial\gamma_{13}} + \gamma_{22}\frac{\partial}{\partial\gamma_{23}} + 2\,\gamma_{23}\frac{\partial}{\partial\gamma_{33}} \end{aligned} \tag{5.8}$$

The algebra $g_r$ corresponding to the group $G_r$ has the following non zero commutators:

$$\begin{array}{lll} [X_1, X_2] = X_2, & [X_1, X_3] = X_3, & [X_1, X_4] = -X_4, \\ [X_2, X_4] = X_1 - X_5, & [X_2, X_5] = X_2, & [X_2, X_6] = X_3, \\ [X_3, X_4] = -X_6, & [X_4, X_5] = -X_4, & [X_5, X_6] = X_6 \end{array} \tag{5.9}$$

It is evident from the aforesaid commutators (5.9) that the group is non abelian, hence we cannot diagonalize all generators at the same time. The derived algebra of $g_r$ is

$$g_{r'} = \{[X_A, X_B] : X_A, X_B \in g_r\} \Rightarrow g_{r'} = \{X_1 - X_5, X_3, X_6, \} \tag{5.10}$$

the second derived algebra is

$$g_{r''} = \{[X_A, X_B] : X_A, X_B \in g_{r'}\} \Rightarrow g_{r''} = \{X_3, X_6\} \tag{5.11}$$

and, finally, the third derived algebra is

$$g_{r'''} = \{[X_A, X_B] : X_A, X_B \in g_{r''}\} \Rightarrow g_{r'''} = \{0\} \tag{5.12}$$

Thus, the group $G_r$ is solvable since $g_{r'''}$ is zero.

The relations (5.9) imply that $X_3, X_6$ commute with one another, as well as with the generator $Y_2$, so that we can bring them to their normal form at the same time. To this purpose, the transformation of $\gamma_{\alpha\beta}$ is:

$$\begin{cases} \gamma_{11} = e^{u_1} \\ \gamma_{12} = e^{u_1}\, u_2 \\ \gamma_{13} = e^{u_1}\, (u_2\, u_3 + u_5) \\ \gamma_{22} = e^{u_1}\, u_4 \\ \gamma_{23} = e^{u_1}\, (u_3\, u_4 + u_2\, u_5) \\ \gamma_{33} = e^{u_1}\, \left(-2\, e^{u_6} + u_3^2\, u_4 + 2\, u_2\, u_3\, u_5 + u_5^2\right) \end{cases} \tag{5.13}$$

In these coordinates, the generators $Y_2, X_I$ take the form

$$\begin{aligned} &Y_2 = \frac{\partial}{\partial u_1} \quad X_3 = \frac{\partial}{\partial u_5} \quad X_6 = \frac{\partial}{\partial u_3} \\ &X_4 = 2\, u_2 \frac{\partial}{\partial u_1} + \left(-2\, u_2^2 + u_4\right) \frac{\partial}{\partial u_2} - u_5 \frac{\partial}{\partial u_3} - 2\, u_2\, u_4 \frac{\partial}{\partial u_4} - 2\, u_2 \frac{\partial}{\partial\, u_6} \\ &X_5 = u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3} + 2\, u_4 \frac{\partial}{\partial u_4} \quad X_2 = \frac{\partial}{\partial u_2} + 2\, u_2 \frac{\partial}{\partial u_4} - u_3 \frac{\partial}{\partial u_5} \\ &X_1 = 2 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2} - 2\, u_4 \frac{\partial}{\partial u_4} - u_5 \frac{\partial}{\partial u_5} - 2 \frac{\partial}{\partial u_6} \end{aligned} \tag{5.14}$$

i.e., the functions $u_1$, $u_3$, $u_5$ appear in the field equations only through their derivatives.

► **The solution space in the gauge $N^\alpha = 0$**

Before we begin with the solution of the Einstein equations, we must examine which values are permissible for the functions $u_i, i = 1, \ldots, 6$. The determinant of the matrix $\gamma_{\alpha\beta}$ is:

$$det[\gamma_{\alpha\beta}] = 2\, e^{3\, u_1 + u_6}\, \left(u_2^2 - u_4\right) \tag{5.15}$$

so, the inequality $u_2^2 - u_4 > 0$ must hold.

The two linear constraints $E_1 = 0$, $E_2 = 0$ in the variables (5.13) give:

$$\begin{aligned} E_1 =& 0 \Rightarrow u_2\,\dot{u}_3 + \dot{u}_5 = 0 \\ E_2 =& 0 \Rightarrow u_4\,\dot{u}_3 + u_2\,\dot{u}_5 = 0 \end{aligned} \tag{5.16}$$

The above system for $\dot{u}_3, \dot{u}_5$ has non zero solutions only in the non permissible case $u_2^2 - u_4 = 0$, so that we obtain

$$u_3 = k_3, \quad u_5 = k_5 \tag{5.17}$$

However, these values of $u_3$, $u_5$ render $\gamma_{13}, \gamma_{23}$ functionals depending on $\gamma_{11}, \gamma_{12}, \gamma_{22}$, see (5.13). Thus, it is possible to set $\gamma_{13}, \gamma_{23}$ to zero by a proper automorphism matrix (5.6), so that we have the following theorem:

*The general solution of Bianchi Type V corresponds to a block diagonal form in the basis of the 1-forms $\boldsymbol{\sigma}^\alpha$.*

Having proved that the matrix $\gamma_{\alpha\beta}$ is block diagonal with no loss of generality, we can repeat the former procedure for determining the generators, i.e., starting from the matrix:

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \tag{5.18}$$

with the corresponding automorphism matrix

$$M = \begin{pmatrix} e^{s_1} & s_2 & 0 \\ s_4 & e^{s_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.19}$$

In this case, the three generators $Y_2, X_1 + X_3, X_2$ commute so that they can be diagonalized at the same time. By applying the transformation:

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1 + 2\,u_6 - \frac{1}{2}\,u_4} & e^{u_1 + 2\,u_6 - \frac{1}{2}\,u_4}\,u_2 & 0 \\ e^{u_1 + 2\,u_6 - \frac{1}{2}\,u_4}\,u_2 & e^{u_1 + 2\,u_6 - \frac{1}{2}\,u_4}\,(e^{u_4} + u_2^2) & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{5.20}$$

the generators take the form

$$Y_2 = \frac{\partial}{\partial u_1},\; X_2 = \frac{\partial}{\partial u_2},\; X_1 + X_3 = \frac{\partial}{\partial u_6} \tag{5.21}$$

which means that the field equations are a first order system for the variables $\dot{u}_1$, $\dot{u}_2$ and $\dot{u}_6$. Consequently, the only variable the second derivative of which appears in the field equations is $u_4$. In order to further simplify this situation, we will also use the generator $Y_1 = \frac{\partial}{\partial t}$.

The variable $u_4$ that does not appear in the generators $Y_2$, $X_1$, $X_2$, can be used to parameterize time $t$, i.e., to apply the transformation

$$t \to u_4(t) = s,\ u_1(t) \to u_1(s),\ u_2(t) \to u_2(s),\ u_6(t) \to u_6(s) \tag{5.22}$$

Of course, this transformation can be applied only when the function $u_4(t)$ is not constant. So, we have to examine separately two cases, i.e., $u_4(t)$ being constant and not constant.

## 5.1 The case $u_4(t) = $ const.

Assuming that $u_4(t) = k_4$, the linear constraints $E_1 = 0$, $E_2 = 0$ are identically satisfied. The third linear constraint is transformed as

$$E_3 = 0 \Rightarrow \dot{u}_6 = 0 \Rightarrow u_6 = k_6 \tag{5.23}$$

Substituting in the quadratic constraint $E_0 = 0$ yields the following lapse function:

$$N^2 = \frac{1}{12}\, e^{-k_4+u_1} \left(3\, e^{k_4}\, \dot{u}_1^2 - \dot{u}_2^2\right) \tag{5.24}$$

Using this lapse function in the equations of motion (2.5), we have as unknown functions $u_1$, $u_2$. Returning to $E_{11} = 0$, it is evident that it is proportional to the quantity

$$\dot{u}_2\,(6\, e^{k_4} \dot{u}_1^2\, \dot{u}_2 - 2\, \dot{u}_2^3 + 3\, e^{k_4}\, \dot{u}_2\, \ddot{u}_1 - 3\, e^{k_4}\, \dot{u}_1\, \ddot{u}_2)$$

The vanishing of the quantity in the parentheses leads to a zero lapse, therefore it is rejected. The case $u_2 = k_2$ satisfies all field equations and gives for the lapse function

$$N^2 = \frac{1}{4}\, e^{u_1}\, \dot{u}_1^2 \Rightarrow (N\, \boldsymbol{d}\, t)^2 = \left(\boldsymbol{d}\, e^{u_1/2}\right)^2 \Rightarrow (N\, \boldsymbol{d}\, t)^2 = (\boldsymbol{d}\, \tau)^2 \tag{5.25}$$

Thus, choosing the time gauge $u_1 = 2\, \ln \tau$ and the automorphism matrix

$$M = \begin{pmatrix} e^{k_4/4-k_6} & -e^{-k_4/4-k_6}\, k_2 & 0 \\ 0 & e^{-k_4/4-k_6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.26}$$

we obtain the metric

$$d\, s^2 = -\boldsymbol{d}\, \tau^2 + \tau^2\, (\boldsymbol{\sigma}^1)^2 + \tau^2\, (\boldsymbol{\sigma}^2)^2 + \tau^2\, (\boldsymbol{\sigma}^3)^2 \tag{5.27}$$

describing a **flat** spacetime.

## 5.2 The case $u_4(t) \neq$ const.

In this case, putting $u_4(t) = \ln t^2$ implies that the first two linear constraints are identically zero, while the third linear constraint $E_3 = 0$ yields

$$E_3 = 0 \Rightarrow \dot{u}_6 = 0 \Rightarrow u_6 = k_6 \tag{5.28}$$

The quadratic constraint $E_o = 0$ defines the following lapse function:

$$N^2 = \frac{e^{u_1}}{12\,t^2}\left(3\,t^2\,\dot{u}_1^2 - \dot{u}_2^2 - 1\right) \tag{5.29}$$

The substitution of the aforementioned lapse function $N^2$ and $u_6$ in equation $E_{11} = 0$ has as a result that the coefficient of $\ddot{u}_2$ is proportional to the quantity

$$\dot{u}_2\,(t\,\dot{u}_1 - 1)\,.$$

Consequently, before we solve equation $E_{11} = 0$ for $\ddot{u}_2$, we must secure that the above quantity is not zero. The vanishing of the term $t\,\dot{u}_1 - 1$ leads to metric with signature $(--++)$, hence it is non permissible. However, the vanishing of the term $\dot{u}_2$ leads to a solution of the field equations as follows: Substituting $u_2 = k_2$ in equation $E_{11} = 0$, the latter becomes proportional to

$$(3\,t\,\dot{u}_1 - 1)(3\,t^2\,\ddot{u}_1 - 3\,t^2\,\dot{u}_1^2 + 3\,t\,\dot{u}_1 + 1) \tag{5.30}$$

The vanishing of the first term leads to incompatibility, while the vanishing of the second term gives

$$u_1 = k_{12} + \ln\left(\operatorname{csch}\left(k_{11} + \frac{1}{\sqrt{3}}\,\ln t\right)\right). \tag{5.31}$$

As a result, all field equations are satisfied. Now, using the transformation $t = e^{\sqrt{3}\,(\tau - k_{11})}$, redefining the constant $k_{12} = \ln\kappa^2$, and acting with the automorphism matrix

$$M = \begin{pmatrix} e^{-\frac{\sqrt{3}\,k_{11}}{2} - k_6} & -e^{\frac{\sqrt{3}\,k_{11}}{2} - k_6}\,k_2 & 0 \\ 0 & e^{\frac{\sqrt{3}\,k_{11}}{2} - k_6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.32}$$

we obtain the metric

$$d\,s^2 = \kappa^2\left(\frac{1}{4}\operatorname{csch}^3\tau\,\boldsymbol{d}\,\tau^2 + e^{-\sqrt{3}\,\tau}\,\operatorname{csch}\tau(\boldsymbol{\sigma}^1)^2 + \;e^{\sqrt{3}\,\tau}\,\operatorname{csch}\tau(\boldsymbol{\sigma}^2)^2 + \operatorname{csch}\tau(\boldsymbol{\sigma}^3)^2\right) \tag{5.33}$$

This metric does not admit a homothety, hence the constant $\kappa$ cannot be absorbed. The metric was first derived by Joseph [22].

Since (5.30) does not vanish, we solve equation $E_{11} = 0$ for $\ddot{u}_2$ and, by substituting in $E_{22} = 0$ the latter becomes proportional to the quantity

$$(t\,\dot{u}_2 + (3\,t\,\dot{u}_1 - 1)\,u_2)\left(3\,t^2\,\ddot{u}_1 - 3\,t\,(\dot{u}_2^2 - 1)\,\dot{u}_1 + \dot{u}_2^2 - 3\,t^2\,\dot{u}_1^2 + 1\right) \tag{5.34}$$

The vanishing of the first term

$$u_2 = e^{-3\,u_1}\,k_2\,t \tag{5.35}$$

leads to a solution of the field equations as follows: Solving $E_{11}=0$ for $\ddot{u}_1$ and substituting in $E_{12}=0$ we get $k_2=0$. As a result, equation $E_{11}=0$ yields

$$\ddot{u}_1 = \dot{u}_1^2 - \frac{1}{t}\,\dot{u}_1 - \frac{1}{3\,t^2} \Rightarrow u_1 = k_{12} + \ln\left(\operatorname{sech}\left(k_{11} + \frac{1}{\sqrt{3}}\,\ln t\right)\right) \tag{5.36}$$

that satisfies also the rest of the field equations. Now, using the transformation $t \mapsto e^{-\sqrt{3}\,(\xi+k_{11})}$, $x \mapsto \frac{1}{2}\,(\sqrt{3}\,k_{11} - x)$, redefining the constant $k_{12} = \ln\lambda^2$, and acting with the automorphism matrix

$$M = \begin{pmatrix} 0 & e^{-k_6} & 0 \\ e^{-k_6} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.37}$$

we obtain the metric

$$d\,s^2 = \lambda^2\left(\frac{1}{4}\operatorname{sech}^3\xi\,\boldsymbol{d}\,\xi^2 + e^{-\sqrt{3}\,\xi}\operatorname{sech}\xi\,(\boldsymbol{\sigma}^1)^2 + \; e^{\sqrt{3}\,\xi}\operatorname{sech}\xi\,(\boldsymbol{\sigma}^2)^2 + \operatorname{sech}\xi\,(\boldsymbol{\sigma}^3)^2\right) \tag{5.38}$$

that does not admit a homothety, hence the constant $\lambda$ cannot be absorbed. This metric has a **Euclidean** signature.

After having examined the case of the vanishing of the first term in (5.34), we proceed to the vanishing of the second term and get the following system of differential equations:

$$\ddot{u}_2 = \frac{1}{t}\,(\dot{u}_2^2 + 1)\,\dot{u}_2 \tag{5.39a}$$

$$\ddot{u}_1 = \frac{1}{3\,t^2}\left(3\,t^2\,\dot{u}_1^2 - 3\,t\,\dot{u}_1 + 3\,t\,\dot{u}_1\,\dot{u}_2^2 - \dot{u}_2^2 - 1\right) \tag{5.39b}$$

Equation (5.39a) can be readily integrated since it is of first order for $\dot{u}_2(t)$ or, more easily, if we multiply by $\dot{u}_2$ and put $\dot{u}_2^2(t) = U(t)$, i.e.

$$\begin{aligned}
\dot{u}_2\,\ddot{u}_2 &= \frac{1}{t}\,(\dot{u}_2^2+1)\,\dot{u}_2^2 \Rightarrow \frac{1}{2}\,\dot{U} = \frac{1}{t}\,(U+1)\,U \Rightarrow \\
\frac{d\,U}{U\,(U+1)} &= \frac{2\,d\,t}{t} \Rightarrow \ln\frac{U}{U+1} = 2\,\ln t + 2\,k_{21} \Rightarrow \\
U &= \frac{e^{2\,k_{21}}\,t^2}{1 - e^{2\,k_{21}}\,t^2} \Rightarrow u_2 = k_{22} \pm e^{-k_{21}}\,\sqrt{1 - e^{2\,k_{21}}\,t^2}
\end{aligned} \tag{5.40}$$

Substituting in (5.39b) we have for $u_1$:

$$u_1 = k_{12} - \ln\left(\sinh\left(k_{11} + \frac{1}{\sqrt{3}}\operatorname{arccoth}\sqrt{1 - e^{2\,k_{21}\,t^2}}\right)\right) \tag{5.41}$$

With these results, we unexpectedly obtain again Joseph's metric (5.33).

# 6 Bianchi Type $IV$

For this type the structure constants are

$$\begin{array}{ll} C^1{}_{13} = -C^1{}_{31} = C^2{}_{23} = -C^2{}_{32} = \frac{1}{2} & \\ C^1{}_{23} = -C^1{}_{32} = \frac{1}{2} & \\ C^\alpha{}_{\beta\gamma} = 0 & for\, all\, other\, values\, of\, \alpha\beta\gamma \end{array} \tag{6.1}$$

Using these values in the defining relation (2.2) of the 1-forms $\sigma_i^\alpha$ we obtain

$$\sigma_i^\alpha = \begin{pmatrix} 0 & e^{-x} & x\,e^{-x} \\ 0 & 0 & -e^{-x} \\ 1 & 0 & 0 \end{pmatrix} \tag{6.2}$$

The corresponding vector fields $\xi_\alpha^i$ (satisfying $[\xi_\alpha, \xi_\beta] = C^\gamma{}_{\alpha\beta}\xi_\gamma$) with respect to which the Lie Derivative of the above 1-forms is zero are:

$$\xi_1 = -\partial_y \quad \xi_2 = \partial_z \quad \xi_3 = \partial_x + (y - z)\partial_y + z\partial_z \tag{6.3}$$

The Time Depended A.I.D.'s are described by

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} P(t) & P(t)\ln(c\,P(t)) & x(t) \\ 0 & P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix} \tag{6.4}$$

and

$$P^\alpha \quad = \quad \left(x(t)\left(\ln\frac{x(t)}{P(t)}\right)' - y'(t), y(t)\left(\ln\frac{y(t)}{P(t)}\right)', -(\ln P(t))'\right) \tag{6.5}$$

where $P(t), x(t)$ and $y(t)$ are arbitrary functions of time. As we have already remarked the three arbitrary functions appear in $P^\alpha$ and thus can be used to set the shift vector to zero.

The remaining symmetry of the EFE's is, consequently, described by the constant matrix:

$$M = \begin{pmatrix} e^{s_1} & s_2 & s_3 \\ 0 & e^{s_1} & s_4 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.6}$$

where the parametrization has been chosen so that the matrix becomes identity for the zero value of all parameters.

Thus the induced transformation on the scale factor matrix is $\tilde{\gamma}_{\alpha\beta} = M_\alpha^\mu M_\beta^\nu \gamma_{\mu\nu}$,

which explicitly reads:

$$\begin{cases} \tilde{\gamma}_{11} = e^{2\,s_1}\,\gamma_{11} \\ \tilde{\gamma}_{12} = e^{s_1}\,s_2\,\gamma_{12} + e^{2\,s_1}\,\gamma_{12} \\ \tilde{\gamma}_{13} = e^{s_1}\,\left(s_3\,\gamma_{11} + s_4\,\gamma_{12} + \gamma_{13}\right) \\ \tilde{\gamma}_{22} = e^{2\,s_1}\,\gamma_{22} + 2\,e^{s_1}\,s_2\,\gamma_{12} + s_2^2\,\gamma_{11} \\ \tilde{\gamma}_{23} = e^{s_1}\,\left(s_3\,\gamma_{12} + s_4\,\gamma_{22} + \gamma_{23}\right) + s_2\,\left(s_3\,\gamma_{11} + \gamma_{13} + s_4\,\gamma_{12}\right) \\ \tilde{\gamma}_{33} = {s_3}^2\,\gamma_{11} + s_3\,\left(\gamma_{12} + \gamma_{13}\right) + {s_4}^2\,\gamma_{22} + 2\,s_4\,\left(s_3\,\gamma_{12} + \gamma_{23}\right) + \gamma_{33} \end{cases} \tag{6.7}$$

The previous equations, define a group of transformations $G_r$ of dimension $r = dim(Aut(IV)) = 4$. The four generators of the group are:

$$X_1 = 2\,\gamma_{11}\frac{\partial}{\partial\gamma_{11}} + 2\,\gamma_{12}\frac{\partial}{\partial\gamma_{12}} + \gamma_{13}\frac{\partial}{\partial\gamma_{13}} + 2\,\gamma_{22}\frac{\partial}{\gamma_{22}} + \gamma_{23}\frac{\partial}{\gamma_{23}} \tag{6.8}$$

$$X_2 = \gamma_{11}\frac{\partial}{\partial\gamma_{12}} + 2\,\frac{\partial}{\partial\gamma_{22}} + \gamma_{13}\frac{\partial}{\partial\gamma_{23}} \tag{6.9}$$

$$X_3 = \gamma_{11}\frac{\partial}{\partial\gamma_{13}} + \gamma_{12}\frac{\partial}{\partial\gamma_{23}} + 2\,\gamma_{13}\frac{\partial}{\partial\gamma_{33}} \tag{6.10}$$

$$X_4 = \gamma_{12}\frac{\partial}{\partial\gamma_{13}} + \gamma_{22}\frac{\partial}{\partial\gamma_{23}} + 2\,\gamma_{23}\frac{\partial}{\partial\gamma_{33}} \tag{6.11}$$

The algebra $g_r$ that corresponds to the group $G_r$ has the following table of non-zero commutators:

$$\begin{array}{lll} [X_1, X_2] = 0, & [X_1, X_3] = X_3, & [X_1, X_4] = X_4, \\ [X_2, X_3] = 0, & [X_2, X_4] = X_3, & [X_3, X_4] = 0 \end{array} \tag{6.12}$$

As it is evident from the above commutators (6.12) the group is non-abelian, so we cannot diagonalize at the same time all the generators. However, if we calculate the derived algebra of $g_r$, we have

$$g_{r'} = \{[X_A, X_B] : X_A, X_B \in g_r\} \Rightarrow g_{r'} = \{X_3, X_4\} \tag{6.13}$$

and furthermore, it's second derived algebra reads:

$$g_{r''} = \{[X_A, X_B] : X_A, X_B \in g_{r'}\} \Rightarrow g_{r''} = \{0\} \tag{6.14}$$

Thus, the group $G_r$ is solvable since the $g_{r''}$ is zero. As it is evident $X_3, X_4, Y_2$ generate an Abelian subgroup, and we can, therefore, bring them to their normal form

simultaneously. The appropriate transformation of the dependent variables is:

$$\begin{cases} \gamma_{11} = e^{u_1} \\ \gamma_{12} = e^{u_1}\, u_2 \\ \gamma_{13} = e^{u_1}\, \left(-e^{u_4}\, u_2^2 + u_3 + u_2\, u_5\right) \\ \gamma_{22} = e^{u_1+u_4} \\ \gamma_{23} = e^{u_1}\, \left(u_2\,(u_3-1) + e^{u_4}\, u_5\right) \\ \gamma_{33} = e^{u_1}\, \left(e^{-2\,u_6} + {u_3}^2 - e^{-u_4}\, u_2^2\,(2\,u_3-1) + 2\,u_2\; u_5\,(u_3-1) + e^{u_4}\, u_5^2\right) \end{cases} \tag{6.15}$$

In these coordinates the generators $Y_2, X_A$ assume the form:

$$\begin{aligned} &Y_2 = \tfrac{\partial}{\partial u_1} \quad X_4 = \tfrac{\partial}{\partial u_5} \quad X_3 = \tfrac{\partial}{\partial u_3} \\ &X_2 = \tfrac{\partial}{\partial u_2} + \left(e^{-u_4}\, u_2 - u_5\right) \tfrac{\partial}{\partial u_3} + 2\, e^{-u_4}\, u_2\, \tfrac{\partial}{\partial u_4} + e^{-2\,u_4}\, \left(e^{u_4} - 2\, u_2^2\right) \tfrac{\partial}{\partial u_5} \\ &X_1 = 2\, \tfrac{\partial}{\partial u_1} - u_3\, \tfrac{\partial}{\partial u_3} + \left(e^{-u_4}\, u_2 - u_5\right) \tfrac{\partial}{\partial u_5} + \tfrac{\partial}{\partial u_6} \end{aligned} \tag{6.16}$$

Evidently, a first look at (6.15) gives the feeling that it would be hopeless even to write down the Einstein equation. However, the simple form of the first three of the generators (6.16) ensures us that these equations will be of first order in the functions $\dot{u}_1$, $\dot{u}_3$ and $\dot{u}_5$.

## 6.1 Description of the Solution Space

Before we begin solving the Einstein equations, a few comments for the possible values of the functions $u_i, i = 1, \ldots, 6$ will prove very useful.

The determinant of $\gamma_{\alpha\beta}$, is

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1 - 2\,u_6}\, \left(e^{u_4} - u_2^2\right) \tag{6.17}$$

so we must have $e^{u_4} > u_2^2$ .

The two linear constraint equations, written in the new variables (6.15), give

$$E_1 = 0 \Rightarrow e^{u_4}\, \dot{u}_3 + u_2^2\, \dot{u}_4 + u_2\,(e^{u_4}\, \dot{u}_5 - \dot{u}_2) = 0 \tag{6.18}$$

$$E_2 = 0 \Rightarrow (3\, e^{u_4} + u_2)\, \dot{u}_2 - e^{u_4}\,(1 + 3\, u_2)\, \dot{u}_3 - (u_2 + 3\, e^{u_4})\,(u_2\, \dot{u}_4 + e^{u_4}\, \dot{u}_5) = 0 \tag{6.19}$$

Solving this system for the functions $\dot{u}_3, \dot{u}_5$ we have

$$\dot{u}_3 = 0, \quad \dot{u}_5 = e^{-u_4}\, \left(\dot{u}_2 - u_2\, \dot{u}_4\right) \tag{6.20}$$

yielding to

$$u_3 = k_3, \quad u_5 = k_5 + e^{-u_4}\, u_2 \tag{6.21}$$

Now, these values of $u_3, u_5$ make $\gamma_{13}, \gamma_{23}$ functionally dependent upon $\gamma_{11}, \gamma_{12}, \gamma_{22}$ (see (6.15)). It is thus possible to set these two components to zero by means of an appropriate constant automorphism.

*Without loss of generality, we can start our investigation of the solution space for Type IV vacuum Bianchi Cosmology from a block-diagonal form of the scale-factor matrix (and, of course, zero shift)*

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \tag{6.22}$$

Note that this conclusion could have not been reached off mass-shell, due to the fact that the time-dependent Automorphism (6.4) does not contain the necessary two arbitrary functions of time in the (13) and (23) components (besides the fact that all the freedom in arbitrary functions of time has been used to set the shift to zero). As we have earlier remarked, since the algebra (6.12) is solvable, the remaining (reduced) generators $X_1, X_2$ (corresponding to diagonal constant automorphisms) as well as $Y_2$ continue to define a Lie-Point symmetry of the reduced EFE's and can thus be used for further integration of this system of equations.

The remaining (reduced) automorphism generators are

$$\begin{aligned} X_1 &= \gamma_{11}\,\frac{\partial}{\partial\gamma_{11}} + \gamma_{12}\,\frac{\partial}{\partial\gamma_{12}} + \gamma_{22}\,\frac{\partial}{\partial\gamma_{22}} \\ X_2 &= \gamma_{11}\,\frac{\partial}{\partial\gamma_{12}} + 2\,\gamma_{12}\,\frac{\partial}{\partial\gamma_{22}} \end{aligned} \tag{6.23}$$

The appropriate change of dependent variables which brings these generators -along with $Y_2$- into normal form, is described by the following scale-factor matrix :

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+2\,u_6} & e^{u_1+2\,u_6}\,u_2 & 0 \\ e^{u_1+2\,u_6}\,u_2 & e^{u_1+2\,u_6}\,(u_2^2+u_4) & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{6.24}$$

The generators are now reduced to

$$Y_2 = \frac{\partial}{\partial u_1},\; X_2 = \frac{\partial}{\partial u_2},\; X_1 = \frac{\partial}{\partial u_6} \tag{6.25}$$

indicating that the system will be of first order in the derivatives of these variables. The remaining variable $u_4$ will enter, (along with $\dot{u}_4$, $\ddot{u}_4$) explicitly in the system and is therefore advisable (if not mandatory) to be used as the time parameter, i.e. to effect the change of time coordinate

$$t \to u_4(t) = s,\; u_1(t) \to u_1(t(s)),\; u_2(t) \to u_2(t(s)),\; u_6(t) \to u_6(t(s)). \tag{6.26}$$

This choice of time will of course be valid only if $u_4$ is not a constant. We are thus led to consider two cases according to the constancy or non-constancy of this variable.

### 6.1.1 Case I: $u_4(t) = k_4$

In these variables the first two linear constraint equations are identically satisfied, and the determinant of $\gamma_{\alpha\beta}$, is

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1+4\,u_6}\,k_4$$

so we must have $k_4 > 0$ . The third linear constraint reads

$$E_3 = 0 \Rightarrow \dot{u}_2 + 4\,k_4\,\dot{u}_6 = 0 \Rightarrow u_2 = k_2 - 4\,k_4\,u_6 \tag{6.27}$$

Substituting this value into the quadratic constraint equation $E_0$ we obtain for the lapse function

$$N^2 = \frac{k_4}{12\,k_4+1}\,e^{u_1}\,(3\,\dot{u}_1^2 + 8\,\dot{u}_1\,\dot{u}_6 - 4\,(4\,k_4-1)\,\dot{u}_6^2) \tag{6.28}$$

If we substitute this value of the lapse function into the equations of motion (2.5), we are left with the unknown functions $u_1, u_6$. The strategy we follow is to solve one of (2.5) for a second derivative of some function, say $\ddot{u}_6$ and replace the result into the rest of the equations. In order to do that, we must ensure that the coefficient of $\ddot{u}_6$ does not equals to zero. Looking at $E_{11} = 0$ the coefficient of $\ddot{u}_6$ is proportional to

$$\dot{u}_1\,(\dot{u}_1 + 2\,(4\,k_4+1)\,\dot{u}_6) \Rightarrow \{u_1 = k_1 \quad \text{or} \quad u_1 = k_1 - 2\,u_6 - 8\,k_4\,u_6\}$$

Thus we are forced to examine the above equalities before we solve $E_{11} = 0$, for $\ddot{u}_6$.

The first possibility $u_1 = k_1$ yields an inconsistency, so we are left with the second, i.e.

$$u_1 = k_1 - 2\,u_6 - 8\,k_4\,u_6 \tag{6.29}$$

The above choice satisfies *all* the spatial equations $E_{\alpha\beta} = 0$ and gives the lapse function

$$N^2 = 16\,e^{k_1-2(4\,k_4+1)\,u_6}\,k_4^2\,\dot{u}_6^2 \tag{6.30}$$

Redefining the constants $k_1 = \ln\kappa^2$, $k_4 = \frac{-\mu}{4\,(\mu-1)}$, choosing a time parametrization $u_6 = (\mu-1)\,\ln(t)$, and using the automorphism matrix (6.6) with entries $s_1 = 0$, $s_2 = -k_2$, $s_3 = 0$, $s_4 = 0$ we arrive at the line element

$$\begin{aligned} d\,s^2 = -\mu^2\,(\boldsymbol{d}\,t)^2 + t^{2\,\mu}\,(\boldsymbol{\sigma^1})^2 + 2\,t^{2\,\mu}\,\mu\,\ln t\,\boldsymbol{\sigma^1}\,\boldsymbol{\sigma^2} \\ + t^{2\,\mu}\,\mu\left(\mu\,\ln^2 t - \frac{1}{4\,(\mu-1)}\right)(\boldsymbol{\sigma^2})^2 + t^2\,(\boldsymbol{\sigma^3})^2 \end{aligned} \tag{6.31}$$

with $t > 0$, $0 < \mu < 1$. In the above line element we have dropped the constant $\kappa$ since this line element admits a homothetic vector field

$$H = \partial_t + (\mu\,z - y\,(\mu-1))\,\partial_y - z(\mu-1)\,\partial_z \tag{6.32}$$

Line element (6.31) was first derived by Harvey and Tsoubelis [23] and admits, besides the three killing fields (6.3) three more, namely

$$\xi_4 = e^{-x/\mu}\,\partial_t + \frac{e^{-x/\mu}}{t}\,\partial_x \tag{6.33}$$

$$\xi_5 = y\,e^{-x/\mu}\,\partial_t + \frac{y\,e^{-x/\mu}}{t}\,\partial_x + f_1\,\partial_y + f_2\,\partial_z \tag{6.34}$$

$$\xi_6 = z\,e^{-x/\mu}\,\partial_t + \frac{z\,e^{-x/\mu}}{t}\,\partial_x + f_2\,\partial_y + f_3\,\partial_z \tag{6.35}$$

with

$$\begin{aligned} f_1 =& \frac{\mu\,e^{(2\,\mu-1)x/\mu}\,t^{-2\,\mu+1}}{(-2\,\mu+1)^3}\Big(4\,\mu^2\,(2\,\mu-1)^2\,(\mu-1)\,\ln^2 t \\ & -8\,\mu\,(\mu-1)\,(2\,\mu-1)(-\mu+(2\mu-1)\,x)\,\ln t \\ & -\mu+4\,(\mu-1)\,(\mu+(1-2\,\mu)\,x)^2\Big) \\ f_2 =& \frac{4\,\mu\,(\mu-1)\,e^{(2\,\mu-1)x/\mu}\,t^{-2\,\mu+1}}{(-2\,\mu+1)^2}\big(\mu\,(2\,\mu-1)\,\ln t+\mu+(1-2\,\mu)\,x\big) \\ f_3 =& \frac{4\,\mu\,(\mu-1)\,e^{(2\,\mu-1)x/\mu}\,t^{-2\,\mu+1}}{-2\,\mu+1} \end{aligned} \tag{6.36}$$

There is thus a $G_6$ symmetry group acting (of course, multiply transitively) on each $V_3$ of this metric. However, it is interesting to note that we have not imposed the extra symmetry from the beginning, but rather it emerged as a result of the investigation process.

Having ensured that the coefficient of $\ddot{u}_6$ at $E_{11}=0$ is not zero, we can solve this equation for $\ddot{u}_6$ and insert the result into the rest of the spatial equations. But doing that we end up with a zero lapse, indicating that the only solution for this case is described by the line element (6.31).

### 6.1.2 Case II: $u_4(t) = t$

In this case the determinant of the sale factor matrix equals to

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1+4\,u_6}\,t$$

so we must demand that $t>0$ in order for $\gamma_{\alpha\beta}$ to be positive defined.

The first two linear constraints are identically zero while the third one $E_3=0$ can be used to define the function $u_2$

$$E_3=0 \Rightarrow \dot{u}_2+4\,t\,\dot{u}_6+1=0 \Rightarrow u_2=k_2-t-4\int t\,\dot{u}_6\,dt \tag{6.37}$$

and the quadratic constraint $E_o = 0$ defines the lapse function $N^2$

$$N^2 = \frac{e^{u_1}}{12\,t+1}\left(3\,t\,\dot{u}_1^2 + 8\,t\,\dot{u}_1\,\dot{u}_6 - 4\,t\,(4\,t-1)\,\dot{u}_6^2 - 2\,(4\,t-1)\,\dot{u}_6 + 2\,\dot{u}_1 - 1\right) \tag{6.38}$$

Substituting the above values of the lapse $N^2$ and the function $u_2$ in equation $E_{33} = 0$ we find the coefficient of $\ddot{u}_6$ is proportional to

$$\dot{u}_1\,\left(4\,t\,\dot{u}_1 - (4\,t-1)\,(4\,t\,\dot{u}_6 + 1)\right)$$

a quantity that can be safely regarded different from zero, since it's nihilism leads either to zero lapse or to inconsistency. Thus we can solve $E_{33} = 0$ for $\ddot{u}_6$ and substitute it to $E_{11} = 0$. In order to solve this equation for $\ddot{u}_1$ we must be assured that it's coefficient does not equals to zero. Setting this coefficient equal to zero we arrive to the following equation

$$\dot{u}_1 = 1 + (4\,t-1)\,\dot{u}_6$$

which is unacceptable because it leads to inconsistency. After solving equation $E_{11} = 0$ for $\ddot{u}_1$ we finally arrive to the following polynomial system of first order in $\dot{u}_1$, $\dot{u}_6$

$$\ddot{u}_1 = \langle\dot{u}_1|\,B_1\,|\dot{u}_6\rangle, \quad \ddot{u}_6 = \langle\dot{u}_1|\,B_2\,|\dot{u}_6\rangle \tag{6.39}$$

where we have used the notation $\langle\dot{u}_i| = (1\,\dot{u}_i\,\dot{u}_i^2\,\dot{u}_i^3)$ and $|\dot{u}_i\rangle = \langle\dot{u}_i|^t$ with the $4\times 4$ matrices $B_1$, $B_2$ given by

$$B_1 = \frac{1}{t\,(12\,t+1)}\begin{pmatrix} -4\,t-1 & -32\,t^3+2\,t & -64\,t^3+4\,t & 0 \\ 24\,t^2+1 & 16\,t^2+4\,t & 8\,t^2\,(4\,t+1)\,(12\,t-1) & 0 \\ 12\,t^2-t & -16\,t^2 & 0 & 0 \\ -6\,t^2 & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = \frac{1}{2\,t\,(12\,t+1)}\begin{pmatrix} 12\,t+3 & 144\,t^2+8\,t-6 & 16\,t\,(36\,t^2+7\,t-1) & 16\,t^2\,(48\,t^2+8\,t-1) \\ -4 & -24\,t & -32\,t^2 & 0 \\ -6\,t & -12\,t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Due to the form of $B_1, B_2$ (their components are rational functions of the time $t$), system (6.39) can be partially integrated with the help of the following Lie-Bäcklund transformation

$$\dot{u}_1 = \frac{(48\,t^2+16\,t+1)\,\dot{r}(t) - 2\,(12\,t-1)\,\tan r(t)}{4\,\sqrt{t}\,(12\,t+1)} \tag{6.40a}$$

$$\dot{u}_6 = -\frac{\sqrt{t}\,(12\,t+1)\,\dot{r}(t) + 6\,\sqrt{t}\,\tan r(t) + 24\,t + 2}{8\,t\,(12\,t+1)} \tag{6.40b}$$

yielding the single second order ODE for the function $r(t)$

$$\ddot{r} = \left(\tan r + \sqrt{t}\right)\, \dot{r}^2 - 2\,\frac{(6\,t+1)\,\tan r + 6\,\sqrt{t}}{\sqrt{t}\,(12\,t+1)}\,\dot{r} + \frac{36\,t^2\,\tan^2 r + 36\,\sqrt{t^3}\,\tan r - 12\,t - 1}{\sqrt{t^3}\,(12\,t+1)^2} \tag{6.41}$$

This equation contains all the information concerning the unknown part of the solution space of the Type $IV$ vacuum Cosmology. Unfortunately, it does not posses any Lie-point symmetries that can be used to reduce its order and ultimately solve it. However, its form can be substantially simplified through the use of new dependent and independent variable $(\rho, u(\rho))$ according to $r(t) = \mp \arcsin \frac{u(\rho)}{\sqrt{\rho^2-1}}$, $t = \frac{1}{6\,(\rho-1)}$ thereby obtaining the equation

$$\ddot{u} = \pm \frac{1-\dot{u}^2}{\sqrt{6\,(\rho-1)\,(\rho^2-u^2-1)}} \Rightarrow \ddot{u}^2 = \frac{(1-\dot{u}^2)^2}{6\,(\rho-1)\,(\rho^2-u^2-1)} \tag{6.42}$$

This equation is a special case of the general equation

$$\ddot{u}^2 = \frac{(1-\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \tag{6.43}$$

with the values $\kappa=-6$, $\lambda=6$. The general solution of (6.43) was first given in [24] and can be obtained as follows: We first apply the contact transformation:

$$\begin{aligned} u(\rho) &= -\frac{8}{\lambda}\,y(\xi) + \frac{4\,(2\,\xi-1)}{\lambda}\,y'(\xi) \qquad & \rho &= -\frac{\kappa}{\lambda} + \frac{4}{\lambda}\,y'(\xi) \\ \dot{u}(\rho) &= 2\,\xi - 1 & \ddot{u}(\rho) &= \frac{\lambda}{2\,y''(\xi)} \end{aligned} \tag{6.44}$$

which reduces it to

$$\xi^2\,(\xi-1)^2\,y''^2 = -4y'\,(\xi\,y'-y)^2 + 4\,y'^2\,(\xi\,y'-y) - \frac{\kappa}{2}\,y'^2 + \frac{\kappa^2-\lambda^2}{16}\,y' \tag{6.45}$$

This equation is a special form of the equation SD-Ia, appearing in [25], where a classification of all second order second degree ordinary differential equations was performed. The general solution of (6.45) is obtained with the help of the sixth Painlevé transcendent $w := \mathbf{P_{VI}}(\alpha,\beta,\gamma,\delta)$ and reads:

$$\begin{aligned} y \;=\; & \frac{\xi^2\,(\xi-1)^2}{4\,w\,(w-1)(w-\xi)}\left(w' - \frac{w\,(w-1)}{\xi\,(\xi-1)}\right)^2 \\ & + \frac{1}{8}\,(1 \pm \sqrt{2\,\alpha})^2\,(1-2\,w) - \frac{\beta}{4}\left(1-\frac{2\,\xi}{w}\right) \\ & - \frac{\gamma}{4}\left(1 - \frac{2\,(\xi-1)}{w-1}\right) + \left(\frac{1}{8}-\frac{\delta}{4}\right)\left(1 - \frac{2\,\xi\,(w-1)}{w-\xi}\right) \end{aligned} \tag{6.46}$$

where the sixth Painlevé transcendent $w := \mathbf{P_{VI}}(\alpha, \beta, \gamma, \delta)$ is defined by the ODE:

$$\begin{aligned} w'' &= \frac{1}{2}\left(\frac{1}{w-1}+\frac{1}{w}+\frac{1}{w-\xi}\right) w'^2 - \left(\frac{1}{\xi-1}+\frac{1}{\xi}+\frac{1}{w-\xi}\right) w' \\ &+ \frac{w\,(w-1)\,(w-\xi)}{\xi^2\,(\xi-1)^2}\left(\alpha+\beta\,\frac{\xi}{w^2}+\gamma\,\frac{(\xi-1)}{(w-1)^2}+\delta\,\frac{\xi\,(\xi-1)}{(w-\xi)^2}\right) \end{aligned} \tag{6.47}$$

The values of the parameters $(\alpha, \beta, \gamma, \delta)$ of the Painlevé transcendent, can be obtained from the solution of the following system:

$$\alpha-\beta+\gamma-\delta\pm\sqrt{2\,\alpha}+1 = -\frac{\kappa}{2} \tag{6.48a}$$

$$(\beta+\gamma)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right) = 0 \tag{6.48b}$$

$$(\gamma-\beta)\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right)+\frac{1}{4}\left(\alpha-\beta-\gamma+\delta\pm\sqrt{2\,\alpha}\right)^2 = \frac{\kappa^2-\lambda^2}{16} \tag{6.48c}$$

$$\frac{1}{4}\,(\gamma-\beta)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right)^2+\frac{1}{4}\,(\beta+\gamma)^2\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right) = 0 \tag{6.48d}$$

Inserting in (6.48) the values of $\kappa = -6, \lambda = 6$ for Type $IV$, we have twenty-four solutions (counting multiplicities) of this system. In order for the parameters $(\alpha, \beta, \gamma, \delta)$ to be real numbers we end up only with three possibilities

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, \frac{1}{2}\right) \tag{6.49a}$$

$$(\alpha, \beta, \gamma, \delta) = \left(2+\sqrt{3}, 0, 0, -1\right) \tag{6.49b}$$

$$(\alpha, \beta, \gamma, \delta) = \left(2-\sqrt{3}, 0, 0, -1\right) \tag{6.49c}$$

Gathering all the pieces the final form of the *general* line element describing Bianchi Type $IV$ vacuum Cosmology is

$$\begin{aligned} d\,s^2 &= -\frac{e^{u_1(\xi)}}{4\,\xi\,(\xi-1)}\,(\boldsymbol{d}\,\xi)^2+\sqrt{\xi\,(\xi-1)\,y'(\xi)}\,(\boldsymbol{\sigma}^1)^2+2\,\sqrt{\xi\,(\xi-1)\,y'(\xi)}\,u_2(\xi)\,\boldsymbol{\sigma}^1\,\boldsymbol{\sigma}^2 \\ &+\frac{1}{4}\sqrt{\frac{\xi\,(\xi-1)}{y'(\xi)}}\left(4\,u_2^2(\xi)\,y'(\xi)+1\right)\,(\boldsymbol{\sigma}^2)^2+e^{u_1(\xi)}\,(\boldsymbol{\sigma}^3)^2 \end{aligned} \tag{6.50}$$

where

$$u_1'(\xi) = \frac{1-2\,\xi-2\,y(\xi)}{2\,\xi\,(\xi-1)}, \quad u_2'(\xi) = \frac{y(\xi)}{2\,\xi\,y'(\xi)\,(\xi-1)} \tag{6.51}$$

and $y(\xi)$ is given by (6.46).

# 7 Discussion

The present work completes the first phase of the programm initiated in [10], utilizing the automorphisms of the various Bianchi Cosmological Models as Lie-Point Symmetries of the corresponding Einstein's Field Equations with the aim of uncovering their solution space. The power of the method lies in the fact that it constitutes a semi-algorithm which, if successfully applied, results in the acquisition of the entire space of solutions. This successful implementation had, so far, been carried out in the case of Bianchi Type $III$ ([24]) and $VII_h$ ([26]; while the present communication covers Types $I$, $II$, $IV$ and $V$. In all cases considered where the general solution is expected to have three essential constants ($III$, $IV$ and $VII_h$), it is given in terms of the sixth Painlevé transcendent $\mathbf{P_{VI}}$, along the way with all the known particular solutions. It is noteworthy that these known metrics are rediscovered in a systematic way and without any extra assumption, in contrast to how they were originally obtained. The case of Types $I$, $II$, and $V$ is characteristic: The general solutions, not considered as such at the time of their first derivation and containing 1 or 2 essential constants, were produced with the aid of various simplifying ansatzen in a time scale of half a century; Kasner (1921), Taub (1951), Harrison (1959), Joseph (1966). Here, they are comprehensively re-acquired along with the solutions not attributed to any one else which, to the best of our knowledge, are **new**: The Type $I$ metric (3.32), the Type $II$ metrics (4.28) (Euclidean signature), (4.37), (4.51) (Euclidean signature) and the Type $V$ metrics (5.27) and (5.38) (Euclidean signature). The production of metrics with Euclidean signature may, at first sight, strike as odd; since our staring point is a line element of Lorenzian signature. However, it is made possible by allowing the lapse to be determined through the quadratic constrained equation instead of prescribing it by an *ab initio* choice of time gauge.

In the remaining Bianchi Types $VIII$, $IX$ and the exceptional $VI_h$ the number of existing automorphisms is not sufficient to allow our method to reduce the problem to a single **second** order ODE in one unknown function, but rather to a **third** order one. We strongly suspect it to be an equivalent form of the Chazy type equations. However the task of proving it involves the search for an appropriate Lie-Bäcklund transformation which is highly non-trivial and non-algorithmic. We plan to return if and when there is something concrete to be reported.

Some directions for future work include the application of the method in the presence of matter sources and/or in higher dimensions.

# References


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